Part III
Singular homology

12 Zeroth homology group, path components and reduced homology

Suppose X is a topological space. Let $f\colon \Delta_n \to X$ be a singular simplex in X. Since Δ_n is path-connected, the image $f(\Delta_n)$ is a path-connected subset of X (Lemma 3.15), hence is contained in some path-component X_a of X. In particular, as an element of $C_n(X)$, the chain f belongs to a subgroup $C_n(X_a)$. It follows that the singular chain complex of X is completely determined by the singular chain complexes of its components. To formalize this in precise mathematical terms we first make the following definition.

Suppose $(C_a, d_a)_{a\in\mathcal{A}}$ is a collection of chain complexes. We define their **direct sum** to be the chain complex $(C, d) = \oplus_{a\in\mathcal{A}} C_a$ defined by

$$C_n = \oplus_{a\in\mathcal{A}}(C_a)_n,$$

$$d = \oplus_{a\in\mathcal{A}} d_a.$$

This means that for every family $x = (x_a)_{a\in\mathcal{A}} \in \oplus_{a\in\mathcal{A}}(C_a)_n$ the value of the boundary operator $d(x)$ is the family

$$(d_a(x_a))_{a\in\mathcal{A}} \in \oplus_{a\in\mathcal{A}}(C_a)_{n-1}.$$

The verification of the fact that (C, d) is a chain complex i.e. $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{Z}$ is almost trivial.

For every $a \in A$ there exist obvious natural chain mappings $i_a\colon C_a \to C$ (inclusion) and $p_a\colon C \to C_a$ (projection). Since i_b is an injection in every dimension, C_a can be identified with a subcomplex $i_a(C_a)$ of C in a natural way.

Let $(C_a, d_a)_{a\in\mathcal{A}}$ is a collection of chain complexes, D is a chain complex and suppose $f_a\colon C_a \to D$ is a chain mapping for all $a \in \mathcal{A}$. Then, for every $n \in \mathbb{Z}$ we have a homomorphism and every $a \in \mathcal{A}$ we have a homomorphism $f_n = \sum_{a\in\mathcal{A}}(f_a)_n\colon \oplus_{a\in\mathcal{A}}(C_a)_n \to D$, given by Lemma 8.13. It is easy to verify that the collection $f = \{f_n\}_{n\in\mathbb{Z}}$ is then a chain mapping $f\colon \oplus_{a\in\mathcal{A}} C_a \to D$. We denote it, naturally, by $\sum_{a\in\mathcal{A}} f_n$.

Lemma 12.1. *The operation of taking homology groups preserves direct sums of chain complexes. More precisely suppose the chain complex C is a direct* *sum of the complexes $(C_a)_{a\in\mathcal{A}}$. Then inclusion mappings $i_a\colon C_a \to C$ induce a chain isomorphism*

$$i_* = ((i_a)_*)_{a\in\mathcal{A}}\colon\ \oplus_{a\in\mathcal{A}} H_n(C_a) \to H_n(C)$$

for every $n \in \mathbb{Z}$.

Proof. Exercise. □

Suppose (X, A) is a topological pair. Let $(X_a)_{a\in\mathcal{A}}$ be the set of all path-components of X. For every $a \in \mathcal{A}$ let $A_a = A \cap X_a$. There exists an inclusion of pairs $i_\alpha\colon (X_a, A_a) \to (X, A)$. By Example 10.12 the induced chain mapping $i_\sharp\colon C_n(X_a, A_a) \to C_n(X, A)$ is an injection in every dimension. Hence we can regard $C(X_a, A_a)$ as a subcomplex of $C(X, A)$ in a natural way. It turns out that $C(X, A)$ is actually a direct sum of subcomplexes $C(X_a, A_a)$ (up to an isomorphism).

Proposition 12.2. *Suppose (X, A) is a topological pair. Let $(X_a)_{a\in\mathcal{A}}$ be the set of all path-components of X. For every $a \in \mathcal{A}$ let $A_a = A \cap X_a$. Then the sum of the chain inclusions $(i_a)_\sharp\colon C(X_a, A_a) \to C(X, A)$*

$$i = \sum_{a\in\mathcal{A}} (i_a)\colon\ \oplus_{a\in\mathcal{A}} C(X_a, A_a) \to C(X, A)$$

is a chain isomorphism. In other words $C(X, A)$ is actually (isomorphic to) a direct sum of its subcomplexes $C(X_a, A_a)$.

Proof. It is enough to show that the mapping

$$i_n = \sum_{a\in\mathcal{A}} (i_a)_n\colon\ \oplus_{a\in\mathcal{A}} C_n(X_a, A_a) \to C_n(X, A)$$

is a bijection for all $n \in \mathbb{Z}$.
For $n < 0$ this is obvious, so we can assume that $n \geq 0$.

First we show that i is an injection. Suppose $x = (x_a)_{a\in\mathcal{A}} \in \oplus_{a\in\mathcal{A}} C_n(X_a, A_a)$ is such that $i(x) = 0 \in C_n(X, A)$. By Proposition 8.12 the family $(x_a)_{a\in\mathcal{A}}$ is finitely supported and

$$x = \sum_{a\in\mathcal{A}} x_a.$$

Since this sum is essentially finite we can write it as

$$x = x_1 + \ldots + x_n,$$

where $x_i \in C_n(X_{a_i}, A_{a_i})$ and $X_{a_1}, \ldots, X_{a_n}$ are different path-components of X. Recall that for every topological pair (Z, Y) and every $n \in \mathbb{Z}$ the group $C_n(Y, B)$ is a free group with a basis

$$\{\overline{\sigma} \mid \sigma\colon \Delta_n \to Z \text{ is continuous and } \sigma(\Delta_n) \subsetneq Y\}.$$

Hence for every $i = 1, \ldots, n$ there exists $m_i \in \mathbb{N}$ such that

$$x_i = \sum_{j_i=1}^{m_i} n_{j_i} \overline{\sigma_{j_i}},$$

where $n_{j_i} \in \mathbb{Z}$ and $\sigma_{j_i}\colon \Delta_n \to X_{a_i}$ is a singular simplex in the space X_{a_i} and $\sigma_{j_i} \subsetneq A_{a_1}$. We may assume that for every i singular simplices $\sigma_{1_i}, \sigma_{2_i}, \ldots, \sigma_{m_i}$ are different.
It follows that

$$i(x) = \sum_{i=1}^{n} \sum_{j_i=1}^{m_i} \overline{\sigma_{j_i}},$$

where we identify, as usual, a singular simplex $\sigma_{j_i}\colon \Delta_n \to X_{a_i}$ with a singular simplex $\sigma_{j_i}\colon \Delta_n \to X$. Moreover, none of these simplices is an element of $C_n(A)$. On the other hand we are assuming that this sum i.e. $i(x)$ equals to zero. Since

$$\{\overline{\sigma} \mid \sigma\colon \Delta_n \to X \text{ is continuous and } \sigma(\Delta_n) \subsetneq A\}.$$

is a free subset of the group $C_n(X, A)$, it follows that $n_{j_i} = 0$ for all i and all j_i involved. Hence $x_i = 0$ for all $i = 1, \ldots, n$, thus also $x = 0$. We have shown that i is an injection.

Next we prove that i is a surjection. Suppose

$$x = \sum_{i=1}^{n} n_i \overline{\sigma_i}$$

is an element of $C(X, A)$ represented as a finite sum of different singular simplices $\sigma_i\colon \Delta_n \to X$. Since Δ_n is path-connected, for every $i = 1, \ldots, n$ there exists an index $a_i \in \mathcal{A}$ such that $\sigma_i(\Delta_n)$ is contained entirely in the path-component X_{a_i}. Hence, we can regard σ_i as an element of $C_n(X_{a_i})$. We can rewrite x as

$$i(\sum_{i=1}^{n} n_i \overline{\sigma_i}),$$

where each $\overline{\sigma_i}$ is considered as an element of $C_n(X_{a_i}, A_{a_i})$, which, in its turn, is considered as a subgroup of the direct sum $\oplus_{a \in \mathcal{A}} C_n(X_a, A_a)$. This proves surjectivity. $\square$

Corollary 12.3. *Suppose (X, A) is a topological space let $(X_a)_{a\in\mathcal{A}}$ be the set of all path-components of X. For every $a \in \mathcal{A}$ let $A_a = A \cap X_a$. Then the inclusions $i_a\colon (X_a, A_a) \to (X, A)$ induce an isomorphism*

$$\sum_{a\in\mathcal{A}}((i_a)_*)_n\colon\ \oplus_{a\in\mathcal{A}} H_n(X_a, A_a) \to H_n(X, A)$$

for every $n \in \mathbb{Z}$.

Proof. Follows immediately from two previous results. □

In particular for absolute groups we have an isomorphism

$$\sum_{a\in\mathcal{A}}((i_a)_*)\colon\ \oplus_{a\in\mathcal{A}} H_n(X_a) \to H_n(X)$$

for every $n \in \mathbb{Z}$. It follows that, as far as absolute groups goes, it is enough to study homology groups of path-connected spaces.

Next we compute the 0-th homology group of every space. Let X be a topological space. Define a homomorphism of groups $\varepsilon\colon C_0(X) \to \mathbb{Z}$ by asserting

$$\varepsilon(\sigma) = 1$$

for every singular 0-dimensional simplex $\sigma \in \mathrm{Sing}_0(X)$. The set $\mathrm{Sing}_0(X)$ can be identified with the set of points of X, since $\sigma\colon \Delta_0 \to X$ is completely determined by its image, which is a singleton. This is because Δ_0 is a singleton. Notice that for an arbitrary element $x \in C_0(X)$, represented in the standard basis $\mathrm{Sing}_0(X)$,

$$x = \sum_{i=1}^{n} n_i\sigma_i,$$

we have

$$\varepsilon(x) = \sum_{i=1}^{n} n_i.$$

Now $d_0 = 0$, since $C_{-1}(X) = 0$, so $\mathrm{Ker}\, d_0 = C_0(X)$ and hence

$$H_0(X) = C_0(X)/\operatorname{Im} d_1.$$

Suppose $\sigma \in \mathrm{Sing}_1(X)$ is a basis element of $C_1(X)$. Then

$$\varepsilon(d_1(\sigma)) = \varepsilon(d_1^1\sigma - d_1^0\sigma) = 1 - 1 = 0,$$

because $d_1^1\sigma, d_1^0\sigma$ are both elements of $\mathrm{Sing}_0(X)$.
Since this is true for every free generator of $C_1(X)$, we have that

$$\varepsilon \circ d_1 = 0.$$

This means that $B_0(X) = d_1(C_1(X)) \subset \operatorname{Ker} \varepsilon$, so, by Factorization Theorem 6.6, the homomorphism ε induces the homomorphism

$$\varepsilon_* \colon H_0(X) \to \mathbb{Z}.$$

If $X = \emptyset$ is an empty space, then $H_n(X) = 0$ for all $n \in \mathbb{Z}$, since $C_0(X) = 0$. In particular ε_* is a trivial zero mapping.
Suppose X is not an empty space. Then there exists at least one 0-dimensional singular simplex $\sigma \colon \Delta_0 \to X$. Since $\sigma \in Z_0(X)$ and $\varepsilon(\sigma) = 1$, it follows that, for every $n \in \mathbb{Z}$, we have that

$$\varepsilon_*(n\overline{\sigma}) = n.$$

This proves that whenever $X \neq \emptyset$ is non-empty, the mapping $\varepsilon_* \colon H_0(X) \to \mathbb{Z}$ is a surjection.

Proposition 12.4. *Suppose X is path-connected and non-empty. Then $\varepsilon_* \colon H_0(X) \to \mathbb{Z}$ is an isomorphism.*

Proof. We have already observed that ε_* is surjective. Since it is induced by a mapping $\varepsilon \colon C_0(X) \to \mathbb{Z}$, it is enough, by factorization theorem, to prove that $\operatorname{Ker} \varepsilon = \operatorname{Im} d_1 = B_0(X)$.

The inclusion $\operatorname{Im} d_1 \subset \operatorname{Ker} \varepsilon$ is already proved above for all spaces. Conversely suppose that

$$c = \sum_{i=1}^{k} n_i x_i \in \operatorname{Ker} \varepsilon,$$

for some $k \in \mathbb{N}$, $n_i \in \mathbb{Z}$, $x_i \in X$. Then

$$\sum_{i=1}^{k} n_i = \varepsilon(c) = 0.$$

Fix a point $x \in X$. Since X is path-connected, for every $i = 1, \ldots, k$ there exists a path $f_i \colon I \to X$ from x to x_i, i.e. $f_i(0) = x, f_i(1) = x_i$. Let

$$g = \sum_{i=1}^{k} n_i f_i \in C_1(X).$$

Then

$$d_1(g) = \sum_{i=1}^{k} n_i(x_i - x) = \sum_{i=1}^{k} n_i x_i - \Big(\sum_{i=1}^{k} n_i\Big)x = \sum_{i=1}^{k} n_i x_i = c.$$

Hence $c \in \operatorname{Im} d_1$ and we are done. $\square$

Since $1 \in \mathbb{Z}$ is a generator of the free group $\mathbb{Z}$ and $\varepsilon_*(\overline{\sigma}) = 1$ for every $\sigma \in \mathrm{Sing}_0(X)$, it follows that as a generator of $H_0(X)$ for the path-connected space X we can take a homology class $\overline{\sigma}$ of any fixed 0-dimensional singular simplex $\sigma\colon \Delta_0 \to X$. Since the image of such σ is a point of X, and conversely, every point of X is the image of unique $\sigma \in \mathrm{Sing}_0(X)$, it is natural to identify them. Hence $H_0(X)$ is essentially generated by an equivalence class of a fixed point $x \in X$.

Corollary 12.5. *Suppose X is a topological space. Then $H_0(X)$ is a free abelian group on the set of path components of X.*
If $H_0(X) \cong \mathbb{Z}^n$ for $n \in \mathbb{N}$, then X has exactly n path components.

In particular X is path-connected if and only $H_0(X) \cong \mathbb{Z}$.

Proof. The first assertion follows from the previous proposition and Corollary 12.3. The second follows from the the fact that $\mathbb{Z}^{(A)} \cong \mathbb{Z}^n$ if and only if A has n elements (Lemma 8.17). $\square$

The proof of the previous Corollary, together with Corollary 12.3, shows that as the basis of $H_0(X) \cong \mathbb{Z}^{(A)}$ we can take any collection of points $(x_a)_{a \in A}$ (or their homology classes to be precise), where **exactly one point** is chosen from every path-component of X.

Consider a chain complex C' with $C'_n = 0$ for $n \neq 0$, $C'_0 = \mathbb{Z}$ and all boundary operators zero homomorphism. We will denote this complex simply by $\mathbb{Z}$, slightly abusing the notation. Obviously $H_n(\mathbb{Z}) = 0$ for $n \neq 0$ and $H_0(\mathbb{Z}) = \mathbb{Z}$.
Let C be an arbitrary chain complex. A chain mapping $\varepsilon\colon C \to \mathbb{Z}$ reduces to a single homomorphism $\varepsilon_0\colon C_0 \to \mathbb{Z}$ subject to a single condition

$$\varepsilon_0 \circ d_1 = 0$$

since the diagram

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d, "d_1"] & 0 \arrow[d] \\ C_0 \arrow[r, "\varepsilon_0"] & \mathbb{Z} \end{tikzcd}$$

must be commutative, and $\varepsilon_n\colon C_n \to (\mathbb{Z})_n = 0$ for $n \neq 0$.

A surjective homomorphism $\varepsilon\colon C_0 \to \mathbb{Z}$, where C is a non-negative chain complex, that satisfies the condition

$$\varepsilon \circ d_1 = 0$$

is called **an augmentation** of the complex C. The pair (C, ε) is then called **an augmented chain complex**. Above we have constructed a canonical natural augmentation of the complex $C(X)$ for every non-empty topological space X (notice that for empty space the constructed mapping is not surjective, hence not an augmentation). Any augmentation $\varepsilon\colon C_0 \to \mathbb{Z}$ defines a chain mapping $\varepsilon\colon C \to \mathbb{Z}$, where on the right side we consider $\mathbb{Z}$ a chain complex as above.

Since $\varepsilon\colon C \to \mathbb{Z}$ is a chain mapping, its kernel $\widetilde{C} = \operatorname{Ker}\varepsilon$ is a chain subcomplex of C. Clearly $\widetilde{C}_n = C_n$ for $n \neq 0$ and, since ε is surjective in all dimensions, we have an exact short sequence

$$0 \longrightarrow \widetilde{C} \xrightarrow{i_*} C \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

of chain complex and chain mappings. From the corresponding long exact homology sequence we obtain for $n > 0$ the exact sequence

$$H_{n+1}(\mathbb{Z}) = 0 \longrightarrow H_n(\widetilde{C}) \xrightarrow{i_*} H_n(C) \longrightarrow H_n(\mathbb{Z}) = 0$$

and for $n = 0$ the exact sequence

$$H_1(\mathbb{Z}) = 0 \longrightarrow H_0(\widetilde{C}) \xrightarrow{i_*} H_0(C) \xrightarrow{\varepsilon_*} \mathbb{Z} = H_0(\mathbb{Z}) \longrightarrow 0$$

Since $\mathbb{Z}$ is a free abelian group, it follows from the lemma 11.17 that the last sequence splits. Hence it follows that

$$H_n(\widetilde{C}) = H_n(C) \text{ for } n > 0$$

$$H_0(C) = H_0(\widetilde{C}) \oplus \mathbb{Z} \text{ and}$$

$$H_0(\widetilde{C}) \cong \operatorname{Ker}\varepsilon_*.$$

The last claim follows from the fact that the sequence

$$0 \longrightarrow H_0(\widetilde{C}) \xrightarrow{i} H_0(C) \xrightarrow{\varepsilon_*} \mathbb{Z} \longrightarrow 0$$

is exact, so $i\colon H_0(\widetilde{C} \to H_0(C)$ is an injective mapping with image $\operatorname{Im} i_* = \operatorname{Ker} \varepsilon_*$.

In particular these considerations apply to the non-negative augmented chain complex $C(X)$, where X is non-empty topological space, with augmentation $\varepsilon\colon C_0(X) \to \mathbb{Z}$ defined above. The homology groups $H_n(\widetilde{C(X)})$ of the chain complex $\widetilde{C(X)}$ are called the **reduced singular homology groups** of the space X and are denoted $\widetilde{H}_n(X)$. It follows that

$$\widetilde{H}_n(X) = H_n(X) \text{ if } n \neq 0,$$

$$H_0(X) = \widetilde{H}_0(X) \oplus \mathbb{Z} \text{ and}$$
$$\widetilde{H}_0(X) = \operatorname{Ker} \varepsilon_*.$$

It can be proved that $\widetilde{H}_0(X)$ is also a free abelian group (this is trivial, if you know that any subgroup of a free group is free, but this fact is not exactly elementary to prove). For our purposes the following result will suffice.

Proposition 12.6. *Suppose X is a non-empty topological space. Then $\widetilde{H}_0(X) = 0$ if and only X is path-connected.*

Proof. If X is path-connected, ε_* is an isomorphism, in particular $\widetilde{H}_0(X) = \operatorname{Ker} \varepsilon_* = 0$. Conversly if $\widetilde{H}_0(X) = 0$, then $H_0(X) = \widetilde{H}_0(X) \oplus \mathbb{Z} \cong \mathbb{Z}$, so X is path-connected by the corollary 12.5. □

Notice that for the empty space the reduced homology groups are not defined.

Suppose (C, ε) and (C', ε') are augmented chain complexes. The chain mapping of augmented complexes $f\colon (C, \varepsilon) \to (C', \varepsilon')$ is a chain mapping that commutes with augmentation, i.e. satisfies the equation

$$\varepsilon' \circ f = \varepsilon.$$

In practise this means that f is a chain mapping for which the diagram

$$\begin{tikzcd} C_0 \arrow[dr, "\varepsilon"] \arrow[dd, "f_0"'] & \\ & \mathbb{Z} \\ C_0' \arrow[ur, "\varepsilon'"] & \end{tikzcd}$$

commutes, since in all other dimensions similar diagram must commute trivially. Suppose $x \in \widetilde{C}_n$ i.e. $\varepsilon_n(x) = 0$. Then

$$\varepsilon'_n(f_n(x)) = \varepsilon_n(x) = 0,$$

so $f_n(x) \in \widetilde{C'}$. It follows that f maps a subcomplex $\widetilde{C}$ to a subcomplex $\widetilde{C'}$. Since the restriction $f\colon \widetilde{C} \to \widetilde{C'}$ is also chain mapping, there exists an induced mapping $f_*\colon H_n(\widetilde{C}) \to H_n(\widetilde{C'})$.

Of course for $n \neq 0$, $H_n(\widetilde{C}) = H_n(C)$ and similarly for C', so the only interesting case is $n = 0$ and the only new piece of information is the induced mapping $f_*\colon H_0(\widetilde{C}) \to H_0(\widetilde{C'})$.

Suppose

$$\begin{tikzcd} 0 \arrow[r] & C' \arrow[r, "f"] & C \arrow[r, "g"] & \overline{C} \arrow[r] & 0 \end{tikzcd}$$

is a short exact sequence of chain complexes, where C' and C are augmented and f preserves augmentation. Notice that the complex $\overline{C}$ is not assumed to be augmented, only complexes C' and C are. The diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \widetilde{C'}, \arrow[r, "\widetilde{f}"] \arrow[d] & \widetilde{C} \arrow[r, "g|"] \arrow[d] & \overline{C} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & C' \arrow[r, "f"] \arrow[d, "\varepsilon'"] & C \arrow[r, "g"] \arrow[d, "\varepsilon"] & \overline{C} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\mathrm{id}"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

is a commutative diagram of chain complexes and chain mappings (check!). Moreover all columns in this diagram are exact, as well as the middle and bottom rows. By Proposition 11.11 also the upper row

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{C'} \arrow[r, "f|"] & \widetilde{C} \arrow[r, "g|"] & \overline{C} \arrow[r] & 0 \end{tikzcd}$$

is then short exact sequence of chain complexes and chain mappings. From

the theorem 11.8 it follows that there exists long exact sequence in homology
(12.7)

$$\dots \longrightarrow H_{n+1}(\overline{C}) \xrightarrow{\Delta} H_n(C') \xrightarrow{f_*} H_n(C) \xrightarrow{g_*} H_n(\overline{C}) \xrightarrow{\Delta} H_{n-1}(C') \longrightarrow \dots$$

$$\dots \longrightarrow H_1(\overline{C}) \xrightarrow{\Delta} H_0(\widetilde{C'}) \xrightarrow{f_*} H_0(\widetilde{C}) \xrightarrow{g_*} H_0(\overline{C}) \longrightarrow 0 \qquad ,$$

called **the reduced long homology sequence** of the original exact sequence

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0.$$

This sequence ends at trivial group 0 after dimension 0, since we assume that C' is non-negative, so $H_{-1}(C') = 0$. Notice that this sequence differs from the usual long exact sequence
(12.8)

$$\dots \longrightarrow H_{n+1}(\overline{C}) \xrightarrow{\Delta} H_n(C') \xrightarrow{f_*} H_n(C) \xrightarrow{g_*} H_n(\overline{C}) \xrightarrow{\Delta} H_{n-1}(C') \longrightarrow \dots$$

$$\dots \longrightarrow H_1(\overline{C}) \xrightarrow{\Delta} H_0(C') \xrightarrow{f_*} H_0(C) \xrightarrow{g_*} H_0(\overline{C}) \longrightarrow 0 \qquad ,$$

only in dimension 0. Also the boundary operator $\Delta\colon H_1(\overline{C}) \to H_0(\widetilde{C'})$ from the sequence 12.7 and the boundary operator $\Delta\colon H_1(\overline{C}) \to H_0(C')$ from the sequence 12.8 satisfy the commutative diagram

$$\begin{tikzcd} H_1(\overline{C}) \arrow[r, "\Delta"] \arrow[dr, "\Delta"'] & H_0(\widetilde{C'}) \arrow[d, "i_*"] \\ & H_0(C'), \end{tikzcd}$$

where $i\colon \widetilde{C'} \to C'$ is an inclusion. This follows easily by naturality of long exact sequence (details as an exercise). Recall that $i_*\colon H_0(\widetilde{C'}) \to H_0(C')$ is actually an inclusion and $H_0(\widetilde{C'})$ can be identified with a subgroup $\operatorname{Ker} \varepsilon_*$ of $H_0(C')$. Thus $\Delta\colon H_1(\overline{C}) \to H_0(\widetilde{C'})$ is the same mapping as $\Delta\colon H_1(\overline{C}) \to H_0(C')$, only with restricted image.

Let us apply this constructions to the singular homology. Suppose (X, A) is a topological pair and $A \neq \emptyset$. Then there exist an exact sequence

$$0 \longrightarrow C(A) \xrightarrow{i_\sharp} C(X) \xrightarrow{j_\sharp} C(X, A) \longrightarrow 0.$$

Since both A and X are assumed to be non-empty, the complexes $C(A)$ and $C(X)$ are augmented by the mappings $\varepsilon^X : C(X) \to \mathbb{Z}$ and $\varepsilon^A : C(A) \to \mathbb{Z}$ both defined by $\varepsilon(\sigma) = 1$ for every singular 0-simplex σ. By the definition it follows easily that the inclusion $i_\sharp : C(A) \to C(X)$ commutes with augmentation ε. Indeed, suppose $\sigma : \Delta_0 \to A$ is an element of basis $\mathrm{Sing}_0(A)$. Then $i_\sharp(\sigma)$ is an element of basis $\mathrm{Sing}_0(X)$, so

$$\varepsilon^X(i_\sharp(\sigma)) = 1 = \varepsilon^A(\sigma).$$

This proves that $\varepsilon^X \circ i_\sharp = \varepsilon^A$ for basis elements, hence for all elements, in dimension 0. In all other dimensions augmentation mappings are both zero mappings, so the equation $\varepsilon^X \circ i_\sharp = \varepsilon^A$ is trivial for those dimensions.

Hence, by the general results above, there exists **the reduced long singular homology sequence**

$$\ldots \longrightarrow H_{n+1}(X,A) \xrightarrow{\Delta} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j*} H_n(X,A) \xrightarrow{\Delta} H_{n-1}(A) \longrightarrow \ldots$$

$$\ldots \longrightarrow H_1(X,A) \xrightarrow{\Delta} \widetilde{H}_0(A) \xrightarrow{i_*} \widetilde{H}_0(X) \xrightarrow{j_*} H_0(X,A) \longrightarrow 0$$

of the pair (X, A).

Suppose $f : X \to Y$ is a continuous mapping between topological spaces. It is easy to see, that the chain mapping $f_\sharp : C(X) \to C(Y)$ preserves standard augmentations of $C(X)$ and $C(Y)$. As a consequence, there exists induced homomorphism $f_* : \widetilde{H}_n(X) \to \widetilde{H}_n(Y)$, for all $n \in \mathbb{Z}$. Of course it differs from $f_* : H_n(X) \to H_n(Y)$ only in dimension $n = 0$. Also, if we regard $\widetilde{H}_0(X)$ as a subgroup $\operatorname{Ker}\varepsilon$ of $H_0(X)$ and similarly for Y, we see that $f_* : \widetilde{H}_0(X) \to \widetilde{H}_0(Y)$ is just a restriction of $f_* : H_0(X) \to H_0(Y)$ to subgroups.

Reduced long homology sequence is natural with respect to chain mappings that preserve augmentation. In particular in case of singular homology it is natural with respect to the mappings induced by continuous mappings. The proof

The reduced homology groups are not absolutely necessary to study, but they turn out to be convenient from the technical point of view. In many cases one has to make additional arguments for the cases $n = 0$ or $n = 1$, when using ordinary homology groups $H_n(X)$. The group $H_0(X)$ is a bit different from the other homology groups, because it does not "count" how many 0-dimensional holes the space has. The reduced group $\widetilde{H}_0(X)$ instead does. Hence reduced groups make considerations more "symmetric" and universal. We will see many examples of reducing homology groups making calculations easy.

In the end of this section let us compute the singular homology of a singleton space $X = \{x\}$. It is clear that for every $n \in \mathbb{N}$ there exists exactly one (continuous) mapping $\sigma_n \colon \Delta_n \to X$, so $C_n(X)$ is a free abelian group generated by a single element σ_n, in particular isomorphic to $\mathbb{Z}$, for every $n \in \mathbb{Z}$. Clearly $d^i(\sigma_n) = \sigma_{n-1}$ for all $n \geq 1$ and all $i = 0 \dots, n$, so

$$d(\sigma_n) = \sum_{i=0}^{n} (-1)^n \sigma_{n-1} = \begin{cases} \sigma_{n-1}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

Hence the chain complex $C(X)$ is a sequence

$$\dots \longrightarrow \mathbb{Z} = C_{2m+1} \xrightarrow{0} \mathbb{Z} = C_{2m} \xrightarrow{\mathrm{id}} \mathbb{Z} = C_{2m-1} \longrightarrow \dots \longrightarrow \mathbb{Z} = C_1 \xrightarrow{0} \mathbb{Z} = C_0 \xrightarrow{0} 0.$$

Let $n = 2m+1$ be positive odd integer. Then $d_{n+1} \colon C_{n+1} \to C_n$ is an isomorphism, hence $B_n(C(X)) = \operatorname{Im} d_{n+1} = C_n$. The homomorphism $d_n \colon C_n \to C_{n-1}$ is a zero mapping, so $Z_n(C(X)) = \operatorname{Ker} d_n = C_n$. It follows that

$$H_n(X) = C_n / C_n = 0$$

is a trivial group. Similarly (check!) we obtain the same result for even n, except when $n = 0$. In this special case $H_0(X) \cong \mathbb{Z}$ either directly from the definition and sequence above or since a singleton space is path-connected. For the same reason $\widetilde{H}_0(X) = 0$. For the feature reference we put results obtain in the form of an official Proposition.

Proposition 12.9. *Let $X = \{x\}$ be a singleton space. Then*

$$H_n(X) = 0 \textit{ for } n \neq 0$$

$$H_0(X) \cong \mathbb{Z}$$

$$\widetilde{H}_n(X) = 0 \textit{ for all } n \in \mathbb{Z}.$$

A basis of $H_0(X)$ is a singleton $\{\overline{x}\}$, where $\overline{x}$ is a homology equivalence class of the only singular 0-simplex $x \colon \Delta_0 \to X$.

13 Homotopy axiom

So-called "homotopy axiom" for homology groups asserts that homotopic mappings induce the same homomorphism between homology groups. Precisely put this property states as following.

Proposition 13.1. *Suppose that the mappings* $f, g\colon (X, A) \to (Y, B)$ *of pairs are homotopic as mappings of pairs i.e. there exists a mapping* $F\colon (X \times I, A \times I) \to (Y, B)$ *of pairs for which*

$$F(x, 0) = f(x),$$

$$F(x, 1) = g(x)$$

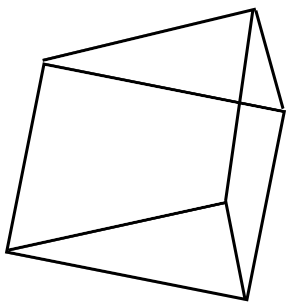
for all $x \in X$. *Then*

$$f_* = g_*\colon H_n(X, A) \to H_n(Y, B), n \in \mathbb{Z}.$$

In the absolute case the same is true for reduced groups.

The rest of this section is devoted to the proof of this result.

Let $f, g\colon (X, A) \to (Y, B)$ be homotopic as mappings of pairs. We also denote this as $f \simeq g$. Let $F\colon (X \times I, A \times I) \to (Y, B)$ be a homotopy between f and g. For a singular simplex $\sigma\colon \Delta_n \to X$ we have a homotopy $F \circ (\sigma \times \mathrm{id})\colon \Delta_n \times I \to Y$ between the mappings $f_\sharp(\sigma)\colon \Delta_n \to X$ and $g_\sharp(\sigma)\colon \Delta_n \to X$. Now, the set $\Delta \times I$ is not a simplex, but it is **a prism**, which is a polyhedron, i.e. can be triangulated in such a way that the bottom and the top (which are both n-simplices) preserve their natural simplicial structure.



To be precise, we let the bottom and the top of this prism to be simplices with vertices $\mathbf{v}_i = (\mathbf{e}_i, 0)$ and $\mathbf{v}'_i = (\mathbf{e}_i, 1)$, $i \in \{0, \ldots, n\}$. It can be verified, that for every index i the sequence $(\mathbf{v}_0, \ldots, \mathbf{v}_i, \mathbf{v}'_i, \ldots, \mathbf{v}'_n)$, is affinely independent, hence span a simplex. Moreover, together these simplices form a simplicial complex, which is a triangulation of $\Delta_n \times I$. We will not prove this and leave this claim as an exercise for the interested reader, since we really don't need this fact, it merely provides us with the motivation and idea for the proof, which will work on the formal algebraic level without verification of the claims above .

Inspired by this for every $\sigma \in \mathrm{Sing}_n(X)$ and for every $i \in \{0, \ldots, n\}$ we define the singular $(n+1)$-simplex $\sigma_i \colon \Delta_{n+1} \to X \times I$ by restricting the mapping $\sigma \times \mathrm{id} \colon \Delta_{n+1} \times I \to X \times I$ on the subset $[\mathbf{v}_0, \ldots, \mathbf{v}_i, \mathbf{v}'_i, \ldots, \mathbf{v}'_n]$. To be more precise, for every $i = 0, \ldots, n$, let $\alpha_n^i \colon \Delta_{n+1} \to \Delta_n \times I$ be the unique affine mapping that maps vertices $(\mathbf{e}_0^{n+1}, \ldots, \mathbf{e}_{n+1}^{n+1})$ of the simplex Δ_{n+1} to the points $(\mathbf{v}_0, \ldots, \mathbf{v}_i, \mathbf{v}'_i, \ldots, \mathbf{v}'_n)$ in that order. In other words α_n^i is characterised by

$$\alpha_n^i(\mathbf{e}_k^{n+1}) = \begin{cases} \mathbf{v}_k, \ 0 \le k \le i, \\ \mathbf{v}'_{k-1}, \ i < k \le n+1. \end{cases} \quad .$$

Such a mapping exists by Lemma 2.15, since the prism $\Delta_n \times I$ is a convex subset of $\mathbb{R}^{n+1}$. Here, as above, $\mathbf{v}_i = (\mathbf{e}_i^n, 0)$ and $\mathbf{v}'_i = (\mathbf{e}_i^n, 1)$.

Recall that for every $n \ge 1$ and every $i = 0, \ldots, n$, we have defined an affine mapping $\varepsilon_n^i \colon \Delta^{n-1} \to \Delta^n$ defined by

$$\varepsilon_n^i(\mathbf{e}_j^{n-1}) = \begin{cases} \mathbf{e}_j^n, \ \text{if } j < i, \\ \mathbf{e}_{j+1}^n, \ \text{if } j \ge i. \end{cases}$$

The useful relationships between mappings α_n^i and ε_n^j are summarized in the following lemmas. Proofs are simple formal calculations, so left as exercises.

Lemma 13.2. *Suppose $n \ge 1$, $i = 0, \ldots, n-1$ and $j = 0, \ldots, n$. Then*

$$(\varepsilon_n^j \times \mathrm{id}) \circ \alpha_{n-1}^i = \begin{cases} \alpha_n^{i+1} \circ \varepsilon_{n+1}^j, 0 \le j \le i, \\ \alpha_n^i \circ \varepsilon_{n+1}^{j+1}, i < j. \end{cases}$$

Proof. Exercise. ☐

Lemma 13.3. *Suppose $n \ge 1$. Then for every $i = 1, \ldots, n$ we have that*

$$\alpha_n^i \circ \varepsilon_{n+1}^i = \alpha_n^{i-1} \circ \varepsilon_{n+1}^i.$$

Proof. Exercise. ☐

Suppose $F\colon X \times I \to Y$ is a homotopy between continuous mappings $f, g\colon X \to Y$. For every $n \in \mathbb{Z}$ we define so-called **prism operator** $P_n\colon C_n(X) \to C_{n+1}(Y)$ as following. For every singular n-simplex $\sigma \in \mathrm{Sing}_n(X)$ we assert

$$P_n(\sigma) = \sum_{i=0}^{n} (-1)^i (F \circ (\sigma \times \mathrm{id}) \circ \alpha_i^n).$$

Then we extend P_n to the unique group homomorphism $P_n\colon C_n(X) \to C_{n+1}(Y)$. Notice that $F \circ (\sigma \times \mathrm{id}) \circ \alpha_i^n$ is a well-defined continuous mapping $\Delta_{n+1} \to Y$, for all i. Hence P_n is well-defined.

For $n < 0$ we naturally assert $P_n\colon C_n(X) \to C_{n+1}(Y)$ to be the zero homomorphism.

Proposition 13.4. *For all $n \in \mathbb{Z}$ the equation*

$$d_{n+1}P_n = g_\sharp - f_\sharp - P_{n-1}d_n$$

holds.

Geometrically we can think of the left side of this equation representing the whole boundary of the prism (the top, the bottom, and horizontal sides), while the right side is the signed sum of the bottom, top and all horizontal sides.

Proof. For $n < 0$ there is nothing to prove, so we may assume that $n \geq 0$. Let σ be a singular n-simplex $\sigma\colon \Delta_n \to X$. It is enough to prove that

$$d_{n+1}P_n(\sigma) = g_\sharp(\sigma) - f_\sharp(\sigma) - P_{n-1}d_n(\sigma).$$

First we calculate the left side. By the definition of the boundary operator we have that

(13.5)

$$d_{n+1}P_n(\sigma) = \sum_{j=0}^{n+1} (-1)^j (P_n(\sigma)) \circ \varepsilon_{n+1}^j) = \sum_{0 \leq i \leq n, 0 \leq j \leq n+1} (-1)^{i+j} F \circ (\sigma \times \mathrm{id}) \circ \alpha_n^i \circ \varepsilon_{n+1}^j.$$

We divide this sum into four pieces. Let

$$A = \sum_{0 \leq j < i \leq n} (-1)^{i+j} F \circ (\sigma \times \mathrm{id}) \circ \alpha_n^i \circ \varepsilon_{n+1}^j,$$

$$B = \sum_{0 \leq j = i \leq n} (-1)^{i+j} F \circ (\sigma \times \mathrm{id}) \circ \alpha_n^i \circ \varepsilon_{n+1}^j,$$

$$C = \sum_{0 \le j = i+1 \le n+1} (-1)^{i+j} F \circ (\sigma \times \mathrm{id}) \circ \alpha_n^i \circ \varepsilon_{n+1}^j,$$

$$D = \sum_{0 \le i+1 < j \le n+1} (-1)^{i+j} F \circ (\sigma \times \mathrm{id}) \circ \alpha_n^i \circ \varepsilon_{n+1}^j.$$

Then $d_{n+1} P_n(\sigma) = A + B + C + D$.

First we show that $B + C = g_\sharp(\sigma) - f_\sharp(\sigma)$. By Lemma 13.3 we have

$$\alpha_n^k \circ \varepsilon_n^k = \alpha_n^{k-1} \circ \varepsilon_n^k \tag{13.6}$$

for every $k = 1, \ldots, n$. In this equation the left-hand side occurs in the sum B, in the term

$$a_k = (-1)^{k+k} F \circ (\sigma \times \mathrm{id}) \circ \alpha_n^k \circ \varepsilon_n^k = F \circ (\sigma \times \mathrm{id}) \circ \alpha_n^k \circ \varepsilon_n^k.$$

Notice that the sign is "positive", since $(-1)^{k+k} = 1$.

On the other hand the right hand side of the equation 13.6 occurs in the sum C, in the term

$$(-1)^{k-1+k} F \circ (\sigma \times \mathrm{id}) \circ \alpha_n^{k-1} \circ \varepsilon_n^k,$$

which, by equation above, equals

$$-a_k = -F \circ (\sigma \times \mathrm{id}) \circ \alpha_n^k \circ \varepsilon_n^k.$$

Notice that minus sign occurs because $(-1)^{k-1+k} = -1$.

It follows that in the sum $B + C$ these terms actually cancel each other out. Do all terms of B and C get cancel out in this fashion? No, not exactly. In the sum B the index pair (i, j) goes through pairs

$$(0,0), (1,1), \ldots, (n,n).$$

In the sum C the index pair (i, j) goes through pairs

$$(0,1), (1,2), \ldots, (n, n+1).$$

Now, the term in B corresponding to $(1,1)$ cancels out with the term in C corresponding to $(0,1)$ (see equation (13.6)!), the term corresponding to $(2,2)$ is paired with the term corresponding to $(1,2)$ and so on.

$$\begin{tikzcd}
(0,0) & (1,1) \arrow[dl] & (2,2) \arrow[dl] & \ldots & (n-1,n-1) \arrow[dl] & (n,n) \arrow[dl] \\
(0,1) & (1,2) & \ldots & (n-2,n-1) & (n-1,n) & (n,n+1)
\end{tikzcd}$$

In the sum B the only term that does not have a "pair" in C, is the term corresponding to indices $i = 0 = j$. That term is

$$F \circ (\sigma \times \mathrm{id}) \circ \alpha_n^0 \circ \varepsilon_n^0 = g \circ \sigma = g_\sharp(\sigma).$$

To verify this first notice that $\alpha_n^0 \circ \varepsilon_n^0(\mathbf{e}_k) = (\mathbf{e}_k, 1)$ for every $k = 0, \ldots, n$. This easily implies that $\alpha_0^n \circ \varepsilon_n^0(x) = (x, 1)$ for all $x \in \Delta_n$. Thus

$$F \circ (\sigma \times \mathrm{id}) \circ \alpha_n^0 \circ \varepsilon_n^0(x) = F(\sigma \times \mathrm{id})(x, 1) = F(\sigma(x), 1) = g(\sigma(x)).$$

In the sum C the only term, which does not have a pair in B is the term corresponding to indices $j = n+1, i = n$. In the same way as above one can easily verify that this term is

$$(-1)^{2n+1} F \circ (\sigma \times \mathrm{id}) \circ \alpha_{n+1}^n \circ \varepsilon_n^n = -f \circ \sigma = -f_\sharp(\sigma).$$

Hence $B + C = g_\sharp(\sigma) - f_\sharp(\sigma)$ and thus

$$d_{n+1} P_n(\sigma) = g_\sharp(\sigma) - f_\sharp(\sigma) + A + D,$$

where

$$A = \sum_{0 \le j < i \le n} (-1)^{i+j} F \circ (\sigma \times \mathrm{id}) \circ \alpha_n^i \circ \varepsilon_n^j,$$
$$D = \sum_{0 \le i+1 < j \le n+1} (-1)^{i+j} F \circ (\sigma \times \mathrm{id}) \circ \alpha_n^i \circ \varepsilon_n^j.$$

To conclude the proof of the proposition, it remains to show that

$$A + D = -P_{n-1} d_n(\sigma).$$

Hence, the next natural step is to calculate $P_{n-1} d_n(\sigma)$. Since P_{n-1} is a homomorphism, we have that

$$P_{n-1}(d_n(\sigma)) = \sum_{j=0}^{n} (-1)^j P_{n-1}(\sigma \circ \varepsilon_n^j).$$

By the definition of P_{n-1} for every $j = 0, \ldots, n$ we have

$$P_{n-1}(\sigma \circ \varepsilon_n^j) = \sum_{i=0}^{n-1} (-1)^i (F \circ (\sigma \circ \varepsilon_n^j) \times \mathrm{id}) \circ \alpha_{n-1}^i.$$

It is trivial to verify that

$$((\sigma \circ \varepsilon_j^n) \times \mathrm{id}) = (\sigma \times \mathrm{id}) \circ (\varepsilon^j \times \mathrm{id}).$$

Hence

$$P_{n-1}(d_n(\sigma)) = \sum_{j=0}^{n}\sum_{i=0}^{n-1}(-1)^{i+j}(F \circ (\sigma \times \mathrm{id}) \circ (\varepsilon^j \times \mathrm{id}) \circ \alpha^i_{n-1}) = A' + D',$$

where

$$A' = \sum_{j \le i \le n-1} (-1)^{i+j}(F \circ (\sigma \times \mathrm{id}) \circ (\varepsilon^j \times \mathrm{id}) \circ \alpha^i_{n-1}),$$

$$D' = \sum_{i<j\le n} (-1)^{i+j}(F \circ (\sigma \times \mathrm{id}) \circ (\varepsilon^j \times \mathrm{id}) \circ \alpha^i_{n-1}).$$

By Lemma 13.2 we have that actually

$$A' = \sum_{j \le i \le n-1} (-1)^{i+j} F \circ (\sigma \times \mathrm{id}) \circ \alpha_n^{i+1} \circ \varepsilon^j_{n+1},$$

$$D' = \sum_{i<j\le n} (-1)^{i+j} F \circ (\sigma \times \mathrm{id}) \circ \alpha_n^{i} \circ \varepsilon^{j+1}_{n+1}.$$

Change of indices $i+1 \mapsto i$ in A' gives

$$A' = \sum_{j<i\le n} (-1)^{i+j+1} F \circ (\sigma \times \mathrm{id}) \circ \alpha_n^{i} \circ \varepsilon^{j}_{n+1} = -A.$$

Similarly one sees that $D' = -D$. Hence

$$d_{n+1}P_n(\sigma) = g_\sharp(\sigma) - f_\sharp(\sigma) + A + D = g_\sharp(\sigma) - f_\sharp(\sigma) - (A' + D') =$$

$$g_\sharp(\sigma) - f_\sharp(\sigma) - P_{n-1}d_n(\sigma).$$

This is exactly what had to be shown. □

Suppose $f, g\colon (X, A) \to (Y, B)$ are mappings of pairs and suppose $F\colon (X \times I, A \times I) \to (Y, B)$ is a homotopy of pairs between f and g.

We have constructed a prism operator $P\colon C_n(X) \to C_{n+1}(Y)$, defined by F for every $n \in \mathbb{Z}$. In the next step we show that it defines an induced prism operator $\overline{P}_n\colon C_n(X, A) \to C_{n+1}(Y, B)$.

Suppose $\sigma \in \mathrm{Sing}_n(A) \subset C_n(A)$, where we think of $C_n(A)$ as a subgroup of $C_n(X)$, as usual. It follows by definitions that in this case $(\sigma \times \mathrm{id}) \circ \alpha_i^n$ is a continuous mapping $\Delta_{n+1} \to A \times I$, for every $i = 0, \ldots, n$. Since $F(A \times I) \subset B$ by our assumption, it follows that $F \circ (\sigma \times \mathrm{id}) \circ \alpha_i^n\colon \Delta_{n+1} \to B$ is a (basis) element of the group $C_{n+1}(B)$. Since P_n is a linear combination of such elements, it follows that the prism operator P_n maps $C_n(A)$ into $C_{n+1}(B)$.

By the standard application of the factorization theorem 6.6 P_n induces a homomorphism $\overline{P}_n\colon C_n(X,A) \to C_{n+1}(Y,B)$ which we call the relative prism operator, for every $n \in \mathbb{Z}$. The formula

$$d_{n+1}P_n = g_\sharp - f_\sharp - P_{n-1}d_n$$

which we have proved in Proposition 13.4 induces the similar formula

$$\overline{d}_{n+1}\overline{P}_n = g_\sharp - f_\sharp - \overline{P}_{n-1}\overline{d}_n$$

for the quotient groups.

The prism operators $\overline{P}_n$ is an example of what is generally known as "chain homotopy".

Definition 13.7. *Suppose $\alpha, \beta\colon C \to C'$ are chain mappings between chain complexes. The collection $H = (H_n)_{n\in\mathbb{N}}$ of homomorphisms $H_n\colon C_n \to C'_{n+1}$ is called* **a chain homotopy** *between α and β if*

$$d'_{n+1}H_n + H_{n-1}d_n = \alpha_n - \beta_n,$$

for all $n \in \mathbb{Z}$.

If for chain mappings $\alpha, \beta\colon C \to C'$ there exists a chain homotopy H between them, we say that α and β are **chain homotopic**.

Notice that chain homotopy **is not** a chain mapping itself!

So far, we have shown the following.

Lemma 13.8. *Suppose $f, g\colon (X,A) \to (Y,B)$ are homotopic as mapping of pairs. Then the induced mappings $f_\sharp, g_\sharp\colon C(X,A) \to C(Y,B)$ are chain homotopic.*

The homotopy axiom 13.1 follows from the previous Lemma and the following general result from homological algebra.

Lemma 13.9. *Suppose $\alpha, \beta\colon C \to C'$ are chain homotopic chain mappings between chain complexes. Then*

$$\alpha_* = \beta_*\colon H_n(C) \to H_n(C')$$

for all $n \in \mathbb{Z}$.

Proof. Let H be a chain homotopy between α and β. Suppose $c \in Z_n(C)$ is a cycle. Then $d(c) = 0$, hence

$$\alpha(c) - \beta(c) = d'H(c) + Hd(c) = d'H(c) \in B_n(C').$$

Hence $\overline{f(c)} = \overline{g(c)}$ in homology. □

The only part of the homotopy axiom 13.1 that remains unproved is the special case of reduced groups. In other words suppose $f, g\colon X \to Y$ are homotopic. We need to show that $f_* = g_*\colon \tilde{H}_n(X) \to \tilde{H}_n(Y)$, $n \in \mathbb{Z}$. For $n \neq 0$ this follows from the homotopy axiom, since then reduced groups are the same as absolute. The only interesting case is the case $n = 0$. But we know that $\widetilde{H}_0(X)$ can be considered as a subgroup of $H_0(X)$ in a natural way, and similarly for Y. Moreover then $f_*\colon \tilde{H}_0(X) \to \tilde{H}_0(Y)$ is just the restriction of $f_*\colon H_0(X) \to H_0(Y)$ to a subgroup and $g_*\colon \tilde{H}_0(X) \to \tilde{H}_0(Y)$ is also a restriction of $g_*\colon H_0(X) \to H_0(Y)$. Since $f_* = g_*$, also their restrictions are.

The Proposition 13.1, which is known as the homotopy axiom for the singular homology theory is proved.

Remark 13.10. *The reader should not be confused by the usage of the word "axiom" in this context. By the axiom one often understands "self-evident truth", which is assumed or otherwise known to be true a priori and requires no proof. We do, of course, prove that the homology groups we have constructed satisfy homotopy axiom, it is not an assumption, it is a property homology groups possess.*

The terminology comes from the axiomatic approach to homology theory. In a nutshell, homotopy axiom is called an axiom, because it is a property one expects any homology theory to satisfy. Besides the singular homology theory there exists a lot of different homology theories in topology. The precise definition of "homology theory" involves a list of "axioms" i.e. properties that homology theory should satisfy, in order to be called a homotopy theory.

Recall that the mapping $f\colon (X, A) \to (Y, B)$ is called **a homotopy equivalence** if there exists $g\colon (Y, B) \to (X, A)$ such that $f \circ g \simeq \mathrm{id}_{(Y,B)}$, $g \circ f \simeq \mathrm{id}_{(X,A)}$ (as mappings of pairs).
Mapping g is then called **a homotopy inverse** of f. The pairs (X, A) and (Y, B) are said to have **the same homotopy type** if there exists homotopy equivalence $f\colon (X, A) \to (Y, B)$.

Corollary 13.11. *Suppose $f\colon (X,A) \to (Y,B)$ is a homotopy equivalence. Then*

$$f_*\colon H_n(X,A) \to H_n(Y,B)$$

is an isomorphism for all $n \in \mathbb{Z}$. The same is true for reduced groups in the absolute case.
In particular spaces of the same homotopy type have isomorphic homology groups, in all dimensions.

Proof. Suppose $g\colon (Y,B) \to (X,A)$ is a homotopy inverse of f. Then $g \circ f \simeq \mathrm{id}$ as mappings $(X,A) \to (X,A)$, so by the homotopy axiom 13.1 we have that

$$g_* \circ f_* = (g \circ f)_* = \mathrm{id}\colon H_n(X,A) \to H_n(X,A) \text{ for all } n \in \mathbb{N}.$$

Similarly $f_* \circ g_* = \mathrm{id}\colon H_n(Y,B) \to H_n(Y,B)$. Hence the homomorphism g_* is the inverse of the homorphism f_*. In particular f_* is an isomorphism. □

The next result is the important generalization of the previous result.

Proposition 13.12. *Suppose $f\colon (X,A) \to (Y,B)$ is a mapping of pairs such that both $f\colon X \to Y$ and $f|A\colon A \to B$ are homotopy equivalences. Then*

$$f_*\colon H_n(X,A) \to H_n(Y,B)$$

is an isomorphism for all $n \in \mathbb{Z}$.

Proof. Exercise (application of the precious result and Five Lemma). □

Example 13.13. *Consider the inclusion of pairs $i\colon (\overline{B}^n, S^{n-1}) \to (\overline{B}^n, \overline{B}^n \setminus \{0\})$. Then $i\colon \overline{B}^n \to \overline{B}^n$ is just the identity mapping, so certainly a homotopy equivalence. The restriction $i\colon S^{n-1} \to \overline{B}^n \setminus \{0\}$ is known to be a homotopy equivalence (Example 7.10, details as exercise). Hence, by the previous proposition the induced mapping*

$$i_*\colon H_n(\overline{B}^n, S^{n-1}) \to H_n(\overline{B}^n, \overline{B}^n \setminus \{0\})$$

is an isomorphism for all $n \in \mathbb{Z}$.

This example also shows that the result 13.12 is truly a generalization of the Corollary 13.11. Namely, it can be shown that as a mapping of pairs the mapping $i\colon (\overline{B}^n, S^{n-1}) \to (\overline{B}^n, \overline{B}^n \setminus \{0\})$ is **not** *a homotopy equivalence, so the result above cannot be obtained by Corollary 13.11.*

Recall that a topological space X is called contractible, if the identity mapping $\mathrm{id}\colon X \to X$ is homotopic to a constant mapping $x_0\colon X \to X$ for some $x_0 \in X$. Precisely this means that there exists a continuous mapping $H\colon X \times I \to X$ such that $H(x,0) = x$, $H(x,1) = x_0$ for all $x \in X$.
If this homotopy is stable at x_0 i.e. $H(x_0,t) = x_0$ for all $t \in I$, we say that the pair (X, x_0) is contractible.

Lemma 13.14. *The space X is contractible if and only if it has the same homotopy type as a singleton space $\{x\}$.*
Similarly the pair (X,x) is contractible if and only if it has the same homotopy type as the pair $(\{x\},\{x\})$.
Every contractible space is path-connected.

Proof. Exercise. $\square$

Example 13.15. *As we already note before the spaces $\mathbb{R}^n$, $\overline{B}^n$ and B^n are contractible. More generally any convex subset C of a finite dimensional vector space V is contractible.*

Example 13.16. *Consider the so-called "topological comb"-space X defined as*

$$X = \bigcup_{n\in\mathbb{N}_+} \{1/n\} \times I \cup \{0\} \times I \cup I \times \{0\}.$$

Then X is contractible. Let $x_0 = (0,1)$. Then the pair (X,x_0) is **not** *contractible. Proofs are left as an exercise.*

Since spaces with the same homotopy type have the same homology and the homology of the singleton space is already calculated, we obtain the following result.

Corollary 13.17. *Suppose X is a contractible space. Then*

$$H_n(X) = 0 \text{ for } n > 0,$$
$$H_0(X) \cong \mathbb{Z},$$
$$\widetilde{H}_0(X) = 0.$$

In particular this is true for $X = \mathbb{R}^n, \Delta_n, \overline{B}^n, B^n$ for all $n \in \mathbb{N}$.

This result seems disappointing at this point. Homology groups of Euclidean spaces turn out to be all isomorphic and quite boring. In particular they alone do not tell us anything about any topological differences of the spaces $\mathbb{R}^n$ for different n. In order to be able to do that we need to learn to calculate homology groups of less trivial, non-contractible spaces, such as the spheres S^n.

14 Excision

So-called excision property is perhaps the most powerful and important property of the singular homology (although in order for it to work we do need other properties, for instance the homotopy axiom). It makes homology groups highly computable and "well-behaved", compared, for instance to the homotopy groups, which do not satisfy excision property.
Formally **Excision axiom** is the following statement

Theorem 14.1. *Suppose $A \subset U \subset X$, where X is a topological space. Suppose $\overline{A} \subset \operatorname{int} U$. Then the inclusion mapping $i\colon (X \setminus A, U \setminus A) \to (X, U)$ of pairs induces isomorphism*

$$i_*\colon H_n(X \setminus A, U \setminus A) \to H_n(X, U)$$

in homology for all $n \in \mathbb{Z}$.

In other words, under the assumptions of the theorem, you can "cut out " or "excite" the set A from the pair (X, U) without altering the homology.

Before proving this theorem let us give an example of its application, which will illuminate its importance and the way this property is applied in practice.

Suppose one wants to calculate the homology groups of the sphere S^n. It is enough to compute the reduced homology groups. Let

$$U = S^n \setminus \{e_{n+1}\} \subset \mathbb{R}^{n+1}.$$

The subset U is homeomorphic to $\mathbb{R}^n$ via stereographic projection from the "north pole", see example 3.8. In particular U is contractible, so its reduced homology groups are trivial, by the results of the previous section (Corollary 13.17). From the long exact reduced homology sequence

$$\tilde{H}_m(U) = 0 \longrightarrow \tilde{H}_m(S^n) \xrightarrow{j_*} H_m(S^n, U) \longrightarrow \tilde{H}_{m-1}(U) = 0$$

we see that the homomorphism $j_*\colon \tilde{H}_m(S^n) \cong H_m(S^n, U)$ is an isomorphism (by exactness), so it is enough to compute the relative groups $H_m(S^n, U)$. Let

$$A = \{\mathbf{x} = (x_1, \ldots, x_{n+1}) \in S^n \mid x_{n+1} < 0\}.$$

Then $\overline{A} = \{x \in S^n \mid x \leq 0\} \subset U = \operatorname{int} U$, so the excision axiom implies that

$$H_m(S^n, U) \cong H_m(S^n \setminus A, U \setminus A).$$

The set $S^n \setminus A$ is a closed upper hemisphere $\{x \in S^n | x_{n+1} \geq 0\}$ of the sphere S^n, which is homeomorphic to the closed ball $\overline{B}^n$ (exercise). Under this homeomorphism the subset $U \setminus A$ corresponds to the punctured ball $\overline{B}^n \setminus \{0\}$. Hence

$$H_m(S^n \setminus A, U \setminus A) \cong H_m(\overline{B}^n, \overline{B}^n \setminus \{0\}).$$

The application of the Proposition 13.12 (see Example 13.13) shows that the inclusion of pairs $(\overline{B}^n, S^{n-1}) \to (\overline{B}^n, \overline{B}^n \setminus \{0\})$ induces an isomorphism

$$H_m(\overline{B}^n, \overline{B}^n \setminus \{0\}) \cong H_m(\overline{B}^n, S^{n-1}),$$

for all $m \in \mathbb{Z}$.

On the other hand the space $\overline{B}^n$ is contractible, so its reduced homology groups are trivial (Corollary 13.17). The part of the long exact reduced homology sequence of the pair $(\overline{B}^n, S^{n-1})$ is the sequence

$$\tilde{H}_m(\overline{B}^n) = 0 \longrightarrow H_m(\overline{B}^n, S^{n-1}) \xrightarrow{\Delta} \tilde{H}_{m-1}(S^{n-1}) \longrightarrow \tilde{H}_{m-1}(\overline{B}^n) = 0.$$

By exactness Δ is an isomorphism, hence

$$H_m(\overline{B}^n, S^{n-1}) \cong \widetilde{H}_{m-1}(S^{n-1}), m \in \mathbb{Z}.$$

Thus we have proved that for all $m \in \mathbb{Z}$ and all $n \geq 1$ we have the equation

$$\widetilde{H}_m(S^n) \cong \widetilde{H}_{m-1}(S^{n-1}).$$

Notice that if we would use ordinary groups instead of reduced, we would have to deal with exceptional cases $m = 0, 1$ and the computations would be more involved, complicated and unsymmetrical. This is a typical illustration of the convenience of reduced groups.

Now we can proceed by induction. The reduced homology groups of the space S^0, which is a discrete space consisting of two points, is easy to calculate directly. They are (exercise)

$$\widetilde{H}_m(S^0) = 0 \text{ for } m \neq 0,$$

$$\widetilde{H}_0(S^0) \cong \mathbb{Z}.$$

Hence the previous computations imply by induction the following important result (and our first interesting example of non-trivial homology groups).

Theorem 14.2. *Singular homology groups of the sphere S^n, $n \in \mathbb{N}$ are the following.*

$$H_m(S^n) = \begin{cases} \mathbb{Z}, & \text{if } m = n \neq 0 \text{ or } n \neq 0, m = 0, \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{if } m = n = 0, \\ 0, & \text{otherwise} \end{cases}.$$

The reduced groups are

$$\tilde{H}_m(S^n) = \begin{cases} \mathbb{Z}, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases}.$$

Since $0 = H_n(S^m) \neq H_n(S^n) = \mathbb{Z}$, $n \neq m$, the Proposition 13.11 immediately implies the following result.

Corollary 14.3. *If $n \neq m$ spheres S^n and S^m don't have the same homotopy type. In particular they are not homeomorphic. A sphere S^n is not contractible for any $n \in \mathbb{N}$.*

Now we can also deduce the long promised classical result.

Corollary 14.4. *Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic if $n \neq m$.*

Proof. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a homeomorphism. By composing it with a translation, if necessary, we may assume that $f(\mathbf{0}) = \mathbf{0}$. Hence f induces a homeomorphism $\mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}^m \setminus \{\mathbf{0}\}$. In particular these spaces have the same homotopy type. But $\mathbb{R}^n \setminus \{\mathbf{0}\}$ has the same homotopy type as S^{n-1}, so we obtain that S^{n-1} and S^{m-1} have the same homotopy type. This contradicts previous corollary. $\square$

This result can be slightly generalized. If $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are both open, non-empty and there is a homeomorphism $f\colon U \to V$, then $n = m$. We leave the proof of this result to the reader. Later we will get to even even stronger claim, which is officially known as "the invariance of domain". It asserts that if $U \subset \mathbb{R}^n$ is open and $f\colon U \to \mathbb{R}^m$ is an injective continuous mapping, then $f(U)$ is open in $\mathbb{R}^m$ and $m = n$.

Another result we can immediately prove using Theorem 14.2, is that S^{n-1} is not a retract of $\overline{B}^n$. A continuous mapping $p\colon X \to A$ is called a **retraction** if A is a subspace of X and $p|A = \mathrm{id}_A$. In other words if $i\colon A \to X$ denotes the inclusion, p is retraction if and only if $p \circ i = \mathrm{id}_A$. If $p\colon X \to A$ is a retraction, we say that A is a **retract** of X.

Corollary 14.5. *S^{n-1} is not a retract of $\overline{B}^n$.*

Proof. Suppose $p\colon \overline{B}^n \to S^{n-1}$ is such that $p \circ i = \mathrm{id}_A$. This implies in particular that

$$p_* \circ i_* = \mathrm{id}\colon \widetilde{H}_{n-1}(S^{n-1}) \to \widetilde{H}_{n-1}(S^{n-1}).$$

It follows that $i_*\colon \widetilde{H}_{n-1}(S^{n-1}) \to \widetilde{H}_{n-1}(\overline{B}^n)$ is an injection. This is however not possible, since $\widetilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z} \neq 0$, while $\widetilde{H}_{n-1}(\overline{B}^n) = 0$. $\square$

It remains to actually prove the excision property. The proof is rather long and tedious. In fact we will prove more general result stated below in the theorem 14.6.

Suppose $\mathcal{A} = (A_i)_{i \in I}$ is a collection of subsets of a topological space X. By $C_n^{\mathcal{A}}(X)$ we denote the free subgroup of $C_n(X)$ generated by exactly those singular simplices $\sigma\colon \Delta_n \to X$ that have the property $\sigma(\Delta_n) \subset A_i$ for some $i \in I$. It is elementary to check that the collection of these subgroups form a chain subcomplex $C^{\mathcal{A}}(X)$ of the chain complex $C(X)$. The homology groups of this chain complex are denoted $H_n^{\mathcal{A}}(X)$.

Theorem 14.6. *Suppose $\mathcal{U}$ is* **a covering** *of X with the property that the collection $\{\mathrm{int}\, U | U \in \mathcal{U}\}$ of interiors of all the elements of $\mathcal{U}$ (with respect to X) is also a covering of X. Then the inclusion mapping $i\colon C^{\mathcal{U}}(X) \to C(X)$ induces isomorphisms*

$$i_*\colon H_n^{\mathcal{U}}(X) \cong H_n(X)$$

in homology for every $n \in \mathbb{Z}$.

Let us first check that the this theorem implies excision property. Suppose $A \subset U \subset X$ are such that $\overline{A} \subset \mathrm{int}\, U$. Denote $V = X \setminus A$ and let $\mathcal{U} = \{U, V\}$. Then (by the general topology)

$$\mathrm{int}\, V = \mathrm{int}(X \setminus A) = X \setminus \overline{A}, \text{ thus}$$

$$\mathrm{int}\, V \cup \mathrm{int}\, U = X.$$

Hence the covering $\mathcal{U}$ satisfies conditions of Theorem 14.6. Also in this case

$$C_n^{\mathcal{U}}(X) = C_n(U) + C_n(V) = \{u + v \mid u \in C_n(U) + C_n(V)\},$$

so it is natural to denote the complex $C^{\mathcal{U}}(X)$ by $C(U) + C(V)$. Both $C(U)$ and $C(V)$ are subcomplexes of $C(U) + C(V)$. Hence also the quotient subcomplexes $(C(V) + C(U))/C(U)$ and $(C(V) + C(U))/C(V)$ exist. Moreover, there exists commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C(U) \arrow[r] \arrow[d, "\mathrm{id}"] & C(V)+C(U) \arrow[r] \arrow[d, "i"] & (C(V)+C(U))/C(U) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C(U) \arrow[r] & C(X) \arrow[r] & C(X,U) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. By the naturality of the long homology sequence we obtain a commutative diagram

$$\begin{tikzcd}
H_n(U) \arrow[r] \arrow[d, "\mathrm{id}"] & H_n^{\mathcal{U}}(X) \arrow[r] \arrow[d, "i_*"] & H_n((C(V)+C(U))/C(U)) \arrow[r] \arrow[d] & H_{n-1}(U) \arrow[r] \arrow[d, "\mathrm{id}"] & H_{n-1}^{\mathcal{U}}(X) \arrow[d, "i_*"] \\
H_n(U) \arrow[r] & H_n(X) \arrow[r] & H_n(X,U) \arrow[r] & H_{n-1}(U) \arrow[r] & H_{n-1}(X).
\end{tikzcd}$$

with exact rows. By Theorem 14.6 the homomorphism i_* is isomorphism for all $n \in \mathbb{Z}$. Also the identity mapping $\mathrm{id}\colon H_n(U) \to H_n(U)$ is trivially isomorphism. By the five-lemma 11.14 it follows that also induced mapping $H_n((C(V)+C(U))/C(U)) \to H_n(X,U)$ is an isomorphism, for all $n \in \mathbb{Z}$.

Next consider a mapping $j\colon C(V)/C(U \cap V) \to (C(V)+C(U))/C(U)$ induced by the inclusion $C(V) \hookrightarrow C(V)+C(U)$. Notice that $C(U\cap V) = C(U) \cap C(V)$. By **the second isomorphism theorem of the group theory** (see example 7.11) j is a (chain) isomorphism. In particular it induces isomorphisms between homology groups.
Collecting all these data together gives us the isomorphism $H_n(V, V \cap U) \cong H_n(X,U)$ induced by the inclusion, for all $n \in \mathbb{N}$. Since $V = X \setminus A$ and $V \cap U = U \setminus A$, this is precisely the excision axiom.

Hence it remains to prove the theorem 14.6. We prove it by showing that $i\colon C^{\mathcal{U}}(X) \to C(X)$ is a **chain homotopy equivalence** i.e. there is a chain mapping $j\colon C(X) \to C^{\mathcal{U}}(X)$ such that $j \circ i$ and $i \circ j$ are chain homotopic to the identity mapping. Since chain homotopic mappings induce the same homorphisms in homology (Lemma 13.9), it follows that

$$j_* \circ i_* = \mathrm{id}, i_* \circ j_* = \mathrm{id},$$

so i_* is indeed an isomorphism.

The construction of j and the homotopies involved is done in several steps.

Suppose V is a finite-dimensional vector space and $D \subset V$ is a convex non-empty subset. Denote by $LC_n(D)$ a subgroup of $C_n(D)$ generated by singular

n-simplices $f\colon \Delta_n \to D$, which are affine as mappings. Notice that such a mapping is uniquely determined by the $(n+1)$-tuple $\{f(\mathbf{e}_0), \dots, f(\mathbf{e}_n)\} \in D^{n+1}$. Conversely, since D is convex, any $(n+1)$-tuple $(\mathbf{f}_0, \dots, \mathbf{f}_n)$ i.e. any element of the cartesian product D^{n+1} defines a unique convex mapping $f\colon \Delta_n \to D$ with the property $f(\mathbf{e}_i) = \mathbf{f}_i$, for all $i = 0, \dots, n$. Thus we might as well define LD_n as a free group generated by the cartesian product D^{n+1}. The boundary operator d of $C_n(D)$ maps LD_n to LD_{n-1} and is defined by the formula (on generators)

$$d(\mathbf{f}_0, \dots, \mathbf{f}_n) = \sum_{i=0}^{n} (-1)^i (\mathbf{f}_0, \dots, \widehat{\mathbf{f}_i}, \dots \mathbf{f}_n).$$

This means that the subgroups $LC_n(D)$ form a chain subcomplex $LC(D)$ of the singular chain complex $C(D)$. Notice that $LC_0(D) = C_0(D)$, so the chain complex $LC(D)$ had a natural augmentation $\varepsilon\colon LC_0(D) \to \mathbb{Z}$ defined by $\varepsilon(\mathbf{f}) = 1$ for all $\mathbf{f} \in D$.

Let $\mathbf{b} \in D$ be a fixed point. We define, for every $n \in \mathbb{N}$, a homomorphism

$$B_{\mathbf{b}} = B\colon LC_n(D) \to LC_{n+1}(D)$$

by

$$B(\mathbf{d}_0, \dots, \mathbf{d}_n) = (\mathbf{b}, \mathbf{d}_0, \dots, \mathbf{d}_n).$$

For $n < 0$ we define $B\colon LC_n(D) \to LC_{n+1}(D)$ to be an obvious zero mapping. Straightforward calculation shows (exercise) that for all $x \in LC_n(D)$ we have that

$$(d_{n+1}B + Bd_n)(x) = \begin{cases} x, & \text{if } n > 0, \\ x - \varepsilon(x)\mathbf{b}, & \text{if } n = 0. \end{cases} \tag{14.7}$$

This implies that the restriction of B to the reduced subgroup

$$\widetilde{LC}_n(D) = \operatorname{Ker} \varepsilon \cap LC_n(D)$$

is a chain homotopy between the identity mapping of $\widetilde{LC}(D)$ and the zero mapping. Chain homotopic mappings induce the same mappings in homology, by Lemma 13.9. On the other hand the identity mapping induces the identity mapping in homology and zero mapping induced zero mapping in homology. The only instance in which the identity mapping of the group equals to the zero mapping, is when the group is a trivial group. Hence $H_n(\widetilde{LC}(D)) = 0$ for all $n \in \mathbb{N}$ i.e. the reduced complex $\widetilde{LC}(D)$ is acyclic.

Next we define so-called **subdivision homomorphism** $S_n\colon LC_n(D) \to LC_n(D)$ by induction on n. For $n = 0$ we assert $S_0 = \mathrm{id}$.

Suppose $n > 0$ and S_{n-1} is already defined. Let $f = (\mathbf{f}_0, \ldots, \mathbf{f}_n)$ be a generator of the group $LC_n(D)$, which by definition is an affine mapping $f\colon \Delta_n \to D$. Let $\mathbf{b}$ be the barycentre of the simplex Δ_n. Let

$$\mathbf{b}_f = f(\mathbf{b}).$$

By $B = B_f\colon LC_{n-1}(D) \to LC_n(D)$ we denote the homomorphism $B_{f(\mathbf{b})}$ defined, as above, by

$$B_f(\mathbf{f}_0, \ldots, \mathbf{f}_{n-1}) = (\mathbf{b}_f, \mathbf{f}_0, \ldots, \mathbf{f}_{n-1}).$$

We define

$$S_n(f) = B_f(S_{n-1}(d_n f))$$

and extent S_n to the unique homomorphism $LC_n(D) \to LC_n(D)$. This makes sense, since $d_n f \in LC_{n-1}(D)$ and $S_{n-1}(d_n f) \in LC_{n-1}(D)$ is already by inductive assumption defined. This concludes the inductive step in the construction of the family (S_n) of homomorphisms. As usual we can define $S_n\colon LC_n(D) \to LC_n(D)$ also for negative $n < 0$ as zero homomorphisms.

Lemma 14.8. *Suppose $f \in LC_n(D)$ is an affine singular simplex $f\colon \Delta_n \to D$. Let $K = K(\Delta_n)$ be the simplicial complex that consists of Δ_n and all its faces. Let K' be the first barycentric subdivision of K. Let*

$$\mathcal{A} = \{f(\sigma) \mid \sigma \in K'\}.$$

Then $S_n(f)$ is an element of $C^{\mathcal{A}}(D)$.

Proof. Since the operator S_n is defined by induction on n, we also prove this claim by induction on n. For $n = 0$ $K' = K$ and $S_0(f) = f$, so the claim is trivial.

Suppose the claim is true for $n - 1$. By the definition

$$d_n f = \sum_{i=0}^{n} (-1)^i d_n^i f,$$

where $d_n^i f = f \circ \varepsilon_n^i\colon \Delta_{n-1} \to D$. Applying inductive assumption on $f_i = d_n^i f$, we see that $S_{n-1} f_i$ is a finite linear sum of the singular affine $(n-1)$-simplices whose images lie in the subset of D of the form

$$f_i(\tau) = f(\varepsilon_n^i(\tau)),$$

where τ is a simplex from the first barycentric division of Δ_{n-1}. Now, recall how barycentric division K' of an arbitrary simplicial complex K is defined. By definition simplicies of K' are simplices spanned by the affinely independent sequences of the form

$$(\mathbf{b}(\sigma_0), \mathbf{b}(\sigma_1), \dots, \mathbf{b}(\sigma_n)),$$

where $\sigma_0 < \sigma_1 < \ldots < \sigma_n$ are simplices of K and $\mathbf{b}(\sigma)$ is a barycentre of σ. Since the mapping $\varepsilon_n^i \colon \Delta_{n-1} \to \Delta_n$ is a simplicial embedding, it follows that it maps any simplex τ of $K'(\Delta_{n-1})$ into the simplex $\varepsilon_n^i(\tau) = \sigma$ of the barycentric division $K'(\Delta_n)$, that lies on the boundary $\operatorname{Bd}\Delta_n$, i.e. is actually a simplex of the barycentric division $K'(\operatorname{Bd}\Delta_n)$ of the boundary of Δ_n. By definition

$$S_n(f) = B_f(S_{n-1}(d_n f)) = \sum_{i=0}^{n} (-1)^i B_f(S_{n-1}(f_i)),$$

so it enough to conclude now that $B_f(S_{n-1}(f_i)) \in C_n(\mathcal{A})$. We have already shown that $S_{n-1}f_i$ is a linear sum of the affine simplices of the form $f(\tau)$, where τ is a simplex from the first barycentric division of the boundary $\operatorname{Bd}\Delta_n$ of Δ_n. Hence it is enough to show that $B_f(f\tau)) \in C_n^{\mathcal{A}}(D)$. This follows straight from the definition of B_f and the definition of the barycentric division $K'(\Delta_n)$. $\square$

The next step is the construction of the chain homotopy $H_n \colon LC_n(D) \to LC_{n+1}(D)$ between the chain mappings S and id. At this point careful reader might notice that we did not actually show that S is a chain mapping. However it turns out that the sheer existence of the chain homotopy H already implies that S is chain mapping, so we do not even need to prove it. More precisely the following statement is true.

Lemma 14.9. *Suppose C and C' are chain complexes. Suppose for every $n \in \mathbb{Z}$ homomorphisms $f_n \colon C_n \to C'_n$ and $f_n \colon C_n \to C'_n$ are given. Suppose also that for every $n \in \mathbb{Z}$ there exists a group homomorphism $F_n \colon C_n \to C'_{n+1}$ such that*

$$d'_{n+1}F_n + F_{n-1}d_n = f_n - g_n$$

for all $n \in \mathbb{Z}$. Then $f = (f_n)$ is a chain mapping if and only if $g = (g_n)$ is a chain mapping.

Proof. By the symmetry it is enough to prove that if g is a chain mapping, then also f is. Let $n \in \mathbb{Z}$. Then

$$d'_n f_n = d'_n(d'_{n+1}F_n + F_{n-1}d_n + g_n) = d'_n d'_{n+1}F_n + d'_n F_{n-1}d_n + d'_n g_n.$$

Here $d'_n d'_{n+1} = 0$ since C' is a chain complex and

$$d'_n g_n = g_{n-1} d_n,$$

since we are assuming g to be a chain mapping. Hence

$$d'_n f_n = (d'_n F_{n-1} + g_{n-1}) d_n = (f_{n-1} - F_{n-2} d_{n-1}) d_n = f_{n-1} d_n,$$

what is what had to be shown. Here we used the equation

$$d'_n F_{n-1} + F_{n-2} d_{n-1} = f_{n-1} - g_{n-1}$$

which is a part of assumption and the fact that $d_{n-1} d_n = 0$, which holds since C is a chain complex. □

The identity mapping $\mathrm{id}\colon C \to C$ is always a chain mapping. Hence, if we manage to construct the family of homomorphisms $H_n\colon LC_n(D) \to LC_{n+1}(D)$ with the property

$$d_{n+1} H_n + H_{n-1} d_n = \mathrm{id} - S_n,$$

the previous lemma would automatically imply that S is a chain mapping.

We construct $H_n\colon LC_n(D) \to LC_{n+1}(D)$ by induction on n. For $n = 0$ we assert $H_0 = 0$. For $n > 0$ we assert

$$H_n(f) = B_f(f - H_{n-1} df)$$

for every singular affine simplex $f\colon \Delta_n \to D$ and then extend H_n to a homomorphism.

Next we check by induction on $n \geq 0$ that $d_{n+1} H_n + H_{n-1} d_n = \mathrm{id} - S_n$ (for $n < 0$ this formula is trivially true). For $n = 0$ this is clear, since $H_0 = H_{-1} = \mathrm{id} - S_0 = 0$. Assume that the formula is true for $n - 1 \geq 0$ i.e.

$$d_n H_{n-1} = \mathrm{id} - S_{n-1} - H_{n-2} d_{n-1}.$$

Then for $n > 0$ we have

$$d_{n+1} H_n(f) = d_{n+1}(B_f(f - H_{n-1} d_n f)) = f - H_{n-1} d_n f - B_f(d_n(f - H_{n-1} d_n f)),\ \text{since}$$

$$d_{n+1} B_f = \mathrm{id} - B_f(d_n f),$$

which was shown before (see 14.7). On the other hand by induction we have that

$$d_n H_{n-1} = \mathrm{id} - S - H_{n-2} d_{n-1},$$

so

$$d_n(f - H_{n-1}d_nf) = d_nf - (d_nf - S_{n-1}(d_nf) - H_{n-2}d_{n-1}d_nf) = S_{n-1}(d_nf).$$

Also $B_f(Sd_nf) = S_n(f)$ by the inductive definition of S.

Hence

$$d_{n+1}H_n(f) = f - H_{n-1}d_nf - B_f(S_{n-1}(d_nf)) = f - S_n(f) - H_{n-1}d_n(f),$$

which is what had to be shown.

So far everything was done for a special case of the convex set D. Next step is to generalize the constructions we have done to arbitrary topological space. Let X be a topological space. We define **barycentric subdivision operator** $S_n\colon C_n(X) \to C_n(X)$ for every $n \in \mathbb{Z}$ as following. For the singular n-simplex $\sigma \in \mathrm{Sing}_n(X)$ we assert

$$S_n(\sigma) = \sigma_\sharp S_n(\mathrm{id}_{\Delta_n}).$$

This is to be understood in the following way. The simplex Δ_n is a convex subset of a finite-dimensional vector space $\mathbb{R}^{n+1}$, so for it we have the operator $S_n\colon LC_n(\Delta_n) \to LC_n(\Delta_n)$ already defined. Moreover the identity mapping $\mathrm{id}\colon \Delta_n \to \Delta_n$ is affine, hence is an element of $LC_n(\Delta_n)$. we apply $S_n\colon LC_n(\Delta_n) \to LC_n(\Delta_n)$ to this element, obtaining an element of $LC_n(\Delta_n)$, which is a subgroup of the group $C_n(\Delta_n)$ of regular n-chains. Finally we apply homomorphism $\sigma_\sharp\colon C_n(\Delta_n) \to C_n(X)$, induced by the continuous mapping $\sigma\colon C_n(\Delta_n) \to C_n(X)$.

By Lemma 14.8 $S_n(\mathrm{id}_{\Delta_n})$ is an element of the group $C^{\mathcal{A}}(\Delta_n)$, where $\mathcal{A}$ is simply a collection of all simplices in the first barycentric subdivision of the simplex Δ_n. Hence, for an arbitrary singular n-simplex $\sigma\colon \Delta_n \to X$, in an arbitrary topological space, $S_n(\sigma)$ is an element of the subgroup $C^{\mathcal{B}}(X)$, where

$$\mathcal{B} = \{\sigma(\tau) \mid \tau \in \mathcal{A}\}$$

In other words $S_n(\sigma)$ is a linear combination of the restrictions of the continuous mapping $\sigma\colon \Delta_n \to X$ to the simplices in the first barycentric division of Δ_n.

Since $S\colon C_n(X) \to C_n(X)$ maps $C_n(X)$ into itself, we can "iterate it" i.e. form compositions of S with itself any number of times, obtaining a homomorphism

$$S^m = \underbrace{S \circ S \circ \ldots \circ S}_{m \text{ times}}.$$

Previous observation, generalized by induction, easily implies the following important result, which is the key to the proof of excision theorem.

Lemma 14.10. *Suppose X is a topological space and $\sigma\colon \Delta_n \to X$ is a singular n-simplex in X. Let $m \in \mathbb{N}$ and denote*

$$\mathcal{B}_m = \{\sigma(\tau) \mid \tau \in K^m\},$$

where K^m is the m-th barycentric subdivision of $K(\Delta_n)$. Then $S^m(\sigma)$ belongs to a subgroup $C^{\mathcal{B}_m}(X)$.

After S_n is defined, we define, for every $n \in \mathbb{Z}$, a chain homotopy $H_n\colon C_n(X) \to C_{n+1}X$ by

$$H_n(\sigma) = \sigma_\sharp(H_n(\mathrm{id}_{\Delta_n})),$$

where again $H_n(\mathrm{id}_{\Delta_n})$ is the image of $\mathrm{id}\colon \Delta_n \to \Delta_n$ under already defined $H_n\colon LC_n(\Delta_n) \to LC_{n+1}(\Delta_n) \subset C_n(\Delta_n)$. The corresponding property of this homotopy shows, that H is then in general a chain homotopy between id and S (exercise). Lemma 14.9 then shows that S is indeed a chain mapping.

We are ready to prove the theorem 14.6.

Suppose $\mathcal{U}$ is a covering of X such that

$$\operatorname{int}\mathcal{U} = \{\operatorname{int} U | U \in \mathcal{U}\}$$

is also a covering of X.

Let $\sigma \in \mathrm{Sing}_n(X)$ be a singular n-simplex in the topological space X. Then, since σ is continuous, the collection

$$\sigma^{-1}(\operatorname{int}\mathcal{U}) = \{\sigma^{-1}(\operatorname{int} U) \mid U \in \mathcal{U}\}$$

is an open covering of the simplex Δ_n. By Proposition 4.21 there exists $m \in \mathbb{N}$ such that the m-th barycentric division K^m of Δ_n is finer than this covering. In particular this means that for every $\tau \in K^m$ there exists $U \in \mathcal{U}$ such that $\tau \subset \sigma^{-1}(\operatorname{int} U)$. It follows that for every $\tau \in K^m$ there exists $U \in \mathcal{U}$ such that

$$\sigma(\tau) \subset \operatorname{int} U \subset U.$$

By Lemma 14.10 this implies, that the iterated barycentric subdivision operator

$$S^m = \underbrace{S \circ \ldots \circ S}_{m \text{ times}}$$

maps the element σ onto an element of the subgroup $C_n^{\mathcal{U}}(X)$.

In the previous observation the index m naturally depends on σ. For every singular n-simplex $\sigma \in \mathrm{Sing}_n(X)$ let $m(\sigma)$ be the **smallest** integer $m(\sigma) \in \mathbb{N}$ that has the property

$$S^{m(\sigma)}(\sigma) \in C_n^{\mathcal{U}}(X).$$

Since S is a chain mapping and $C^{\mathcal{U}}(X)$ is a subcomplex, we have that

$$S_{n-1}^{m(\sigma)}(d_n\sigma) = d_n S_n^{m(\sigma)}(\sigma) \in C_{n-1}^{\mathcal{U}}(X).$$

This implies that for every face $d_n^i(\sigma)$ of the simplex σ we have that

$$m(d_n^i(\sigma)) \leq m(\sigma).$$

Also for any $m \geq m(\sigma)$ evidently $S^m(\sigma) \in C_n^{\mathcal{U}}(X)$, since S maps $C_n^{\mathcal{U}}(X)$ to itself (check!).

It is a straightforward calculation (exercise) to verify that

$$D_m = \sum_{0 \leq i < m} HS^i$$

is a chain homotopy between the chain mappings id and S^m, for every $m \in \mathbb{N}$.

We define $D\colon C_n(X) \to C_{n+1}(X)$ for every $n \in \mathbb{N}$ by asserting

$$D(\sigma) = D_{m(\sigma)}(\sigma).$$

for singular n-simplices σ in X. Now, for a singular n-simplex σ we have that

$$(d_{n+1}D + Dd_n)(\sigma) = (dD_{m(\sigma)}(\sigma) + D_{m(\sigma)}(d\sigma)) - D_{m(\sigma)}(d(\sigma)) + D(d(\sigma)) =$$

$$= (\sigma - S^{m(\sigma)}) - D_{m(\sigma)}(d(\sigma)) + D(d(\sigma)) = \sigma - p(\sigma),$$

where we haveused the equation

$$dD^{m(\sigma)} + D^{m(\sigma)}d = \mathrm{id} - S^{m(\sigma)}$$

and denoted $p(\sigma) = S^{m(\sigma)} + D_{m(\sigma)}(d(\sigma)) - D(d\sigma)$. We claim that $p(\sigma) \in C_n^{\mathcal{U}}(X)$. This is clear for the term $S^{m(\sigma)}$. For all $i \in \{0, \ldots, n\}$ we have that $m(d^i\sigma) \leq m(\sigma)$, so

$$D_{m(\sigma)}(d^i\sigma) - D(d^i\sigma) = D_{m(\sigma)}(d^i\sigma) - D_{m(d^i\sigma)}(d_i\sigma) = \sum_{m(d^i\sigma) \leq j < m(\sigma)} HS^j(d\sigma).$$

Now for $j \geq m(d^i\sigma)$ $S^j(d^i\sigma) \in C_n^{\mathcal{U}}(X)$, as we have already noticed above. Also, it is easy to see that the homotopy H maps $C_n^{\mathcal{U}}(X)$ into itself (check), so the claim is proved.

If we now denote by p the mapping $C(X) \to C^{\mathcal{U}}(X)$ defined by p, we see that D is then a chain homotopy between id and $i \circ p$, where $i\colon C^{\mathcal{U}}(X) \to C(X)$ is an inclusion.
Lemma 14.9 implies that $i \circ p$, hence also p itself, are chain mappings. Moreover, by the definition of p, it follows easily that $p \circ i = \mathrm{id}$, because $m(\sigma) = 0 = m(d^i\sigma)$ for $\sigma \in C^{\mathcal{U}}(X)$, so in that case $S^{m(\sigma)}(\sigma) = \mathrm{id}(\sigma)$, and $D_{m(\sigma)}(d\sigma) = D(d\sigma)$.

Since $p \circ i = \mathrm{id}$, in homology we have that

$$p_* \circ i_* = (p \circ i)_* = \mathrm{id}_* = \mathrm{id}\,.$$

Moreover, since $i \circ p$ is a chain homotopic to the identity mapping, by Lemma 13.9 we obtain that

$$i_* \circ p_* = (i \circ p)_* = \mathrm{id}_* = \mathrm{id}\,.$$

Hence p_* and i_* are inverses of each other, in particular i_* is an isomorphism. This is precisely what Theorem 14.6 claims. Excision property of the singular homology is proved.

Now we know essentially everything one needs to know about the singular homology groups. All the applications and further development of the theory will be done using the properties proved so far, such as excision property and homotopy property.

As a useful example of the calculation that involves different properties of the singular homology we will investigate the exact structure of the group $H_m(\Delta_n, \mathrm{Bd}\,\Delta_n)$ for all $m, n \in \mathbb{N}$.

At this point we can already easily compute these groups *up to an isomorphism.* Indeed we have the long exact reduced homology sequence of the pair $(\Delta_n, \mathrm{Bd}\,\Delta_n)$, which locally looks the following

$$\widetilde{H}_m(\Delta_n) \longrightarrow H_m(\Delta_n, \mathrm{Bd}\,\Delta_n) \xrightarrow{\Delta} \widetilde{H}_{m-1}(\mathrm{Bd}\,\Delta_n) \longrightarrow \widetilde{H}_{m-1}(\Delta_n).$$

The simplex Δ_n is contractible, so its reduced homology groups $\widetilde{H}_m(\Delta_n)$ are trivial 0 groups for all $m \in \mathbb{N}$ (Corollary 13.17). Hence we have the exact sequence

$$0 \longrightarrow H_m(\Delta_n, \mathrm{Bd}\,\Delta_n) \xrightarrow{\Delta} \widetilde{H}_{m-1}(\mathrm{Bd}\,\Delta_n) \longrightarrow 0$$

for all $m \in \mathbb{Z}$. By exactness $\operatorname{Ker}\Delta = \operatorname{Im} 0 = 0$ and $\operatorname{Im}\Delta = \operatorname{Ker} 0 = H_{m-1}(\operatorname{Bd}\Delta_n)$, which means that the homomorphism $\Delta\colon H_m(\Delta_n, \operatorname{Bd}\Delta_n) \to \widetilde{H}_{m-1}(\operatorname{Bd}\Delta_n)$ is an isomorphism for $m \in \mathbb{Z}$. The boundary $\operatorname{Bd}\Delta_n$ is homeomorphic to the sphere S^{n-1} (Corollary 3.21). The reduced homology groups of the sphere S^{n-1} are already calculated in Theorem 14.2. Hence

$$H_m(\Delta_n, \operatorname{Bd}\Delta_n) \cong \begin{cases} \mathbb{Z}, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases} .$$

This calculation tells us in particular that $H_n(\Delta_n, \operatorname{Bd}\Delta_n)$ is isomorphic to the group of integers $\mathbb{Z}$, which is a free group generated by one element. However the calculation does not tell us **which** element of $H_n(\Delta_n, \operatorname{Bd}\Delta_n)$ is its generator.

Consider the singular n-simplex $\mathrm{id}_n = \mathrm{id}\colon \Delta_n \to \Delta_n$, which is an element of the group $C_n(\Delta_n)$. Its boundary is the element

$$d(\mathrm{id}) = \sum_{i=0}^{n} \varepsilon_n^i,$$

which belongs to the subgroup $C_{n-1}(\operatorname{Bd}\Delta_n)$ of the group $C_{n-1}(\operatorname{Bd}\Delta_n)$, since the image of the mapping $\varepsilon_n^i\colon \Delta_{n-1} \to \Delta_n$ lies in the boundary $\operatorname{Bd}\Delta_n$. Hence id is not in general a cycle element in $C(\Delta_n)$ (unless $n = 0$), but its equivalence class in the quotient group $C_n(\Delta_n, \operatorname{Bd}\Delta_n)$ **is** a cycle. Hence there exists a homology class $\overline{\mathrm{id}} \in H_n(\Delta_n, \operatorname{Bd}\Delta_n)$.

Proposition 14.11. *Suppose $n, m \in \mathbb{N}$. Then*

(1) $H_m(\Delta_n, \operatorname{Bd}\Delta_n) = 0$ if $m \neq n$,

(2) the group $H_n(\Delta_n, \operatorname{Bd}\Delta_n)$ is a free abelian group on one element generated by an element $\overline{\mathrm{id}}$.

Proof. The claim (1) is already established, so it is enough to prove the claim (2).

We prove this claim by induction on n by constructing an isomorphism between $\gamma\colon H_n(\Delta_n, d\Delta_n) \to H_{n-1}(\Delta_{n-1}, \operatorname{Bd}\Delta_{n-1})$ that takes id_n to id_{n-1}. Once we have such an isomorphism, it is enough to verify the claim for $n = 0$. But for $n = 0$ the pair $(\Delta_n, \operatorname{Bd}\Delta_n)$ is just a singleton $(\Delta_0, \emptyset) = \Delta_0$. By Proposition 12.9 the group $H_0(\Delta_0)$ is a free abelian group generated by (class of) the unique mapping $\Delta_0 \to \Delta_0$, which is exactly the identity mapping id_0.

It remains to contruct an isomorphism $\gamma\colon H_n(\Delta_n, \operatorname{Bd}\Delta_n) \to H_{n-1}(\Delta_{n-1}, \operatorname{Bd}\Delta_{n-1})$ that takes id_n to id_{n-1} for every $n > 0$.

Denote
$$\Lambda_n^0 = \bigcup_{i>0} d_n^i \Delta_n,$$
which is thus the union of all $(n-1)$-faces of Δ_n, except for the 0'th face. The linear homotopy $\alpha\colon \Delta_n \times I \to \Delta_n$, which contracts Δ_n into the point e_0,
$$\alpha(x,t) = (1-t)x + te_0$$
has the property
$$\alpha(\Lambda_n^0 \times I) \subset \Lambda_n^0$$
(verification left as an exercise). In other words the restriction of $\alpha\colon \Lambda_n^0 \times I \to \Lambda_n^0$ is a contraction of Λ_n^0 into a point. In particular Λ_n^0 is contractible, so its reduced homology groups are all trivial, just as the reduced homology groups of Δ_n. Consider the long exact homology sequence of the pair (Δ_n, Λ_n^0),
$$\ldots \longrightarrow \widetilde{H}_m(\Delta_n) \longrightarrow H_m(\Delta_n, \Lambda_n^0) \longrightarrow \widetilde{H}_{m-1}(\Lambda_n^0) \longrightarrow \ldots.$$
Since all reduced absolute homology groups in this sequence are trivial, we obtain exact sequence of the form
$$0 \longrightarrow H_m(\Delta_n, \Lambda_n^0) \longrightarrow 0$$
for every $m \in \mathbb{Z}$. As usual, by exactness, this implies that $H_m(\Delta_n, \Lambda_n^0) = 0$ for all $m \in \mathbb{Z}$. Next we consider the long exact homology sequence of the triple $(\Delta_n, \operatorname{Bd}\Delta_n, \Lambda_n^0)$, a part of which looks like
$$H_n(\Delta_n, \Lambda_n^0) \xrightarrow{j_*} H_n(\Delta_n, \operatorname{Bd}\Delta_n) \xrightarrow{\Delta} H_{n-1}(\operatorname{Bd}\Delta_n, \Lambda_n^0) \xrightarrow{i_*} H_{n-1}(\Delta_n, \Lambda_n^0).$$
Since $H_n(\Delta_n, \Lambda_n^0) = 0 = H_{n-1}(\Delta_n, \Lambda_n^0)$, we obtain the exactness of the sequence
$$0 \longrightarrow H_n(\Delta_n, \operatorname{Bd}\Delta_n) \xrightarrow{\Delta} H_{n-1}(\operatorname{Bd}\Delta_n, \Lambda_n^0) \longrightarrow 0,$$
which implies that $\Delta\colon H_n(\Delta_n, \operatorname{Bd}\Delta_n) \to H_{n-1}(\operatorname{Bd}\Delta_n, \Lambda_n^0)$ is an isomorphism. To calculate $\Delta(\overline{\mathrm{id}_n}))$ we have to recall how the boundary operator of the long exact homology sequence is defined. First we need an element c of $C_n(\Delta_n, \Lambda_n^0)$ with the property $j(c) = \mathrm{id}_n$. Obviously, the class of the identity mapping id_n, as an element of $C_n(\Delta_n, \Lambda_n^0)$, has this property, so we take it as a c. Next, we take its boundary dc in the complex $C(\Delta_n, \Lambda_n^0)$ and its class,

thought of as an element of $C_{n-1}(\operatorname{Bd}\Delta_n, \Lambda_n^0)$, is an element $\Delta_n(\overline{\mathrm{id}_n})$ we are looking for. In other words

$$\Delta(\overline{\mathrm{id}_n}) = \overline{d(\mathrm{id}_n)}.$$

But in the group $C_{n-1}(\operatorname{Bd}\Delta_n, \Lambda_n^0)$

$$d(\mathrm{id}_n) = \sum_{i=0}^{n} d_i(id_n) = d_0(\mathrm{id}_n),$$

since $d_i \,\mathrm{id}_n \in C_{n-1}(\Lambda_n^0)$ for $i > 0$. Thus finally we obtain, for the isomorpism $\Delta\colon H_n(\Delta_n, \operatorname{Bd}\Delta_n) \to H_{n-1}(\operatorname{Bd}\Delta_n, \Lambda_n^0)$, that

$$\Delta(\overline{\mathrm{id}_n}) = \overline{d_0(\mathrm{id}_n)}.$$

Recall the mapping $\varepsilon_n^0\colon \Delta_{n-1} \to \Delta_n$ which embeds Δ_{n-1} as a 0'th face of Δ_n. For our purposes we think of it is a mapping of pairs $\varepsilon_n^0\colon (\Delta_{n-1}, \operatorname{Bd}\Delta_{n-1}) \to (\operatorname{Bd}\Delta_n, \Lambda_n^0)$ (an embedding), which by definition has the property

$$\varepsilon_n^0 = \varepsilon_n^0 \circ \mathrm{id}_{n-1} = d_0(\mathrm{id}).$$

Next step is to prove that the induced mapping

$$\varepsilon_*\colon H_{n-1}(\Delta_n, \operatorname{Bd}\Delta_{n-1}) \to H_{n-1}(\operatorname{Bd}\Delta_n, \Lambda_n^0)$$

is an isomorphism. Consider the commutative diagram

$$\begin{tikzcd}
 & H_{n-1}(\operatorname{Bd}\Delta_n, \Lambda_n^0) \\
H_{n-1}(\Delta_n, \operatorname{Bd}\Delta_{n-1}) \arrow[ur, "\varepsilon_*"] \arrow[dr, "\varepsilon_*"] & \\
 & H_{n-1}(\operatorname{Bd}\Delta_n \setminus \{\mathbf{e}_0\}, \Lambda_n^0 \setminus \{\mathbf{e}_0\}) \arrow[uu, "i_*"]
\end{tikzcd} ,$$

where we denote by ε also the mapping of pairs $(\Delta_{n-1}, \operatorname{Bd}\Delta_{n-1}) \to (\operatorname{Bd}\Delta_n \setminus \{\mathbf{e}_0\}, \Lambda_n^0 \setminus \{\mathbf{e}_0\})$ defined by the same formula as ε_n^0. By choosing $A = \{\mathbf{e}_0\}$, $U = \Lambda_n^0$, we see that

$$A = \overline{A} \subset \operatorname{int} U = \{\mathbf{x} \in \operatorname{Bd}\Delta_n | x_0 > 0\},$$

so the inclusion of pairs

$$i\colon (\operatorname{Bd}\Delta_n \setminus \{\mathbf{e}_0\}, \Lambda_n^0 \setminus \{\mathbf{e}_0\}) \to (\operatorname{Bd}\Delta_n, \Lambda_n^0)$$

satisfies the assumptions of the Excision Theorem 14.1. Hence the induced mapping

$$i_*\colon H_{n-1}(\operatorname{Bd}\Delta_n \setminus \{\mathbf{e}_0\}, \Lambda_n^0 \setminus \{\mathbf{e}_0\}) \to H_{n-1}(\operatorname{Bd}\Delta_n, \Lambda_n^0)$$

is an isomorphism. Since the diagram 14 commutes, it follows that

$$\varepsilon_*\colon H_{n-1}(\Delta_n, \operatorname{Bd}\Delta_{n-1}) \to H_{n-1}(\operatorname{Bd}\Delta_n, \Lambda_n^0)$$

is an isomorphism if and only if

$$\varepsilon_*\colon H_{n-1}(\Delta_{n-1}, \operatorname{Bd}\Delta_{n-1}) \to H_{n-1}(\operatorname{Bd}\Delta_n \setminus \{\mathbf{e}_0\}, \Lambda_n^0 \setminus \{\mathbf{e}_0\})$$

is an isomorphism. Define

$$\lambda\colon (\operatorname{Bd}\Delta_n \setminus \{\mathbf{e}_0\}, \Lambda_n^0 \setminus \{\mathbf{e}_0\}) \to (\Delta_{n-1}, \operatorname{Bd}\Delta_{n-1})$$

by the formula

$$\lambda(x_0, \ldots, x_n) = \Big(\frac{x_1}{1-x_0}, \ldots, \frac{x_n}{1-x_0}\Big).$$

We leave it to the reader as an exercise to prove that λ is a well-defined mapping of pairs and *a homotopy inverse* of the mapping

$$\varepsilon\colon (\Delta_{n-1}, \operatorname{Bd}\Delta_{n-1}) \to (\operatorname{Bd}\Delta_n \setminus \{\mathbf{e}_0\}, \Lambda_n^0 \setminus \{\mathbf{e}_0\}).$$

This means that this mapping ε is a homotopy equivalence, hence induced mapping ε_* between homology groups is an isomorphism (Corollary 13.11). By the consideration above this implies that also the mapping

$$\varepsilon_*\colon H_{n-1}(\Delta_n, \operatorname{Bd}\Delta_{n-1}) \to H_{n-1}(\operatorname{Bd}\Delta_n, \Lambda_n^0)$$

is an isomorphism for every $n \in \mathbb{Z}$. Now let

$$\gamma = \varepsilon_*^{-1} \circ \Delta\colon H_n(\Delta_n, d\Delta_n) \to H_{n-1}(\Delta_{n-1}, \operatorname{Bd}\Delta_{n-1}).$$

As a composition of two isomorphisms this mapping is an isomorphism. Moreover

$$\gamma(\overline{\mathrm{id}_n}) = \overline{\mathrm{id}_{n-1}}.$$

Thus the needed isomorphism is constructed and the claim follows by induction. $\square$

Corollary 14.12. *Suppose $n > 0$. Then the group $H_n(\operatorname{Bd}\Delta_{n+1})$ is isomorphic to $\mathbb{Z}$ and generated by the class of an element $d\,\mathrm{id} \in C_n(\operatorname{Bd}\Delta_{n+1})$, where* $\mathrm{id}\colon \Delta_{n+1} \to \Delta_{n+1}$.

Proof. By definition

$$d\,\mathrm{id} = \sum_{i=0}^{n} (-1)^{n+1} \varepsilon^i_{n+1}$$

is indeed an element of the subgroup $C_n(\operatorname{Bd}\Delta_{n+1})$ of $C_n(\Delta_{n+1})$. Moreover, since $dd = 0$, this element is obviously a cycle. Hence the homology class $\overline{d\,\mathrm{id}} \in H_n(\operatorname{Bd}\Delta_{n+1})$ really exists.

To deduce that it is a generator consider the reduced exact homology sequence of the pair $(\Delta_{n+1}, \operatorname{Bd}\Delta_{n+1})$ Since Δ_n is contractible, exactness implies (as usual) that

$$\Delta\colon H_{n+1}(\Delta_{n+1}, \operatorname{Bd}\Delta_{n+1}) \to H_n(\operatorname{Bd}\Delta_{n+1})$$

is an isomorphism. By the previous Proposition [id] is a generator of the group $H_{n+1}(\Delta_{n+1}, \operatorname{Bd}\Delta_{n+1})$. It is easy to see, using the definition of the boundary operator Δ that

$$\Delta(\overline{\mathrm{id}}) = \overline{d\,\mathrm{id}}.$$

The claim is proved. ☐

15 The equivalence of the simplicial and singular homologies

In this section we will prove that in order to calculate the singular homology groups of a compact polyhedron pair, it is enough to calculate its simplicial homology, since both are actually isomorphic. More precisely, we will prove the following result

Theorem 15.1. *Suppose (K, L) is a pair of finite Δ-complexes. Then the inclusion $\iota\colon C(K, L) \to C(|K|, |L|)$ induces isomorhisms in homology i.e.*

$$\iota_*\colon H_n(K, L) \to H_n(|K|, |L|)$$

for all $n \in \mathbb{Z}$.

This result has both practical and theoretical applications. Later we will also prove slightly more abstract generalization of this result for the CW-complexes.

Before we go through the proof of Theorem 15.1, let us recall how the mapping $\iota\colon C(K, L) \to C(|K|, |L|)$ is defined exactly. Let K be a Δ-complex. The group $C_n(K)$ of simplicial n-chains is defined to be a free abelian group on the set of all geometrical n-simplices of K. For every geometrical n-simplex σ of K (recall that this is not a simplex per se, but actually an equivalence class of simplices identified together in K) we let $f_\sigma\colon \Delta_n \to |K|$ to be its characteristic mapping. Then we define $\iota_n\colon C_n(K) \to C_n(|K|)$ to be the unique homomorphism with the property

$$\iota_n(\sigma) = f_\sigma,$$

for every geometrical n-simplex σ of K.

Let L be a subcomplex of K. Then ι_n maps $C_n(L)$ into $C_n(|L|)$, hence defines a mapping $\iota_n\colon C_n(K,L) \to C_n(|K|,|L|)$ between factor groups. This mapping is injective for every $n \in \mathbb{Z}$ and defines a chain embedding $\iota\colon C(K,L) \to C(|K|,|L|)$.

Notice, in particular, that the group $C_n(|K|)$ is usually much "larger" then $C_n(K)$ - it contains, as a basis element, every possible continuous mapping $\Delta_n \to K$, and there is usually really a lot of different and weird continuous mapping. The group $C_n(K)$ is, on the other hand, much smaller, especially when K is finite - it has a finite basis, consisting of very simple, linear mappings. Hence the result 15.1 is indeed non-trivial and useful. Consider, for example the calculation of the simplicial homology groups of such spaces as the Mobius band, or projective plane, that we underwent in the section 9 - directly from the definition. It would be impossible to even dream about the similar "directly from the definition" calculation of the corresponding singular homology groups of these spaces.

The theorem 15.1 is actually true for arbitrary Δ-complex pairs (K,L), also infinite, but the proof of the general version is more involved, so we will concentrate on the finite case only.

The theorem 15.1 follows easily from the following special case with the aid of standard technical tricks from homological algebra. Recall that so-called n'th skeleton K^n of the Δ-complex K is defined to be the subcomplex of K consisting of all its simplices with dimension smaller or equal to an integer n.

Lemma 15.2. *Suppose K is a finite Δ-complex and $n \in \mathbb{N}$. Then the inclusion mapping $\iota\colon C(K^n,K^{n-1}) \to C(|K^n|,|K^{n-1}|)$ induces an isomorphism*

$$\iota_*\colon H_m(K^n,K^{n-1}) \to H_m(|K^n|,|K^{n-1}|)$$

in homology for every $m \in \mathbb{N}$

Before we proof this lemma that us actually see, how Theorem 15.1 can be deduced out of it.

Proof of Theorem 15.1 **assuming Lemma** 15.2:

First we prove the absolute case, i.e. we show that $\iota_*\colon H_m(K) \to H_m(|K|)$ is an isomorphism for all $m \in \mathbb{Z}$ and finite Δ-complexes K.

Since K is finite, $K = K^n$, where K^n is an n-skeleton, for some $n \in \mathbb{N}$, so we prove the claim for K^n by induction on n.

For $n = 0$ the claim follows immediately from Lemma 15.2, since $(K^0, K^{-1}) = (K^0, \emptyset)$. It can also be proved directly from the definition. Indeed, first of all K^0 has only simplices of dimension 0, so the group $C_n(K^0)$ is a trivial zero group when $m \neq 0$, while $C_0(K^0)$ is a free group on the set of geometrical 0-simplices of K. Since all but one groups in the chain complex $C^n(K^0)$ are zero groups, all of its boundary operators must be zero mappings. Simple straightforward calculation implies then that

$$H_m(K^0) = 0, \text{ when } m \neq 0$$

and $H_0(K^0)$ is essentially $C_0(K^0)$, a free group on the set of geometrical 0-simplices of K. Moreover $\iota_*(\sigma)$ is an equivalence class of the constant mapping $f_\sigma\colon \Delta_0 \to \{\sigma\}$, for every geometrical 0-simplex σ in K.

On the other hand $|K^0|$ is a discrete space that consists of isolate points, 0-simplices of K. In particular path-connected components of K^0 are singletons $\{\sigma\}$, where σ is a geometrical 0-simplex of K. By Propositions 12.3 and 12.9 we have that

$$H_m(|K^0|) = 0, \text{ when } m \neq 0$$

and $H_0(|K^0|)$ is a free group generated by the constant mappings $f_\sigma\colon \Delta_0 \to K^0$, $f\sigma(1) = \sigma$, for every geometrical 0-simplex σ (i.e. a vertex) in K. It follows that $\iota_*\colon H_m(K^0) \to H_m(|K^0|)$ is an isomorphism in any case.

Suppose the claim is proved for $(n-1)$ i.e. $\iota_*\colon H_m(K^{n-1}) \to H_m(|K^{n-1}|)$ is an isomorphism for all $m \in \mathbb{Z}$. We have to prove that $\iota_*\colon H_m(K^n) \to H_m(|K^n|)$ is an isomorphism for all $m \in \mathbb{Z}$.

Consider the following diagram of chain complexes and chain mappings

$$\begin{tikzcd}
0 \arrow[r] & C(K^{n-1}) \arrow[r, "i"] \arrow[d, "\iota"] & C(K^n) \arrow[r, "j"] \arrow[d, "\iota"] & C(K^n, K^{n-1}) \arrow[r] \arrow[d, "\iota"] & 0 \\
0 \arrow[r] & C(|K^{n-1}|) \arrow[r, "i"] & C(|K^n|) \arrow[r, "j"] & C(|K^n|, |K^{n-1}|) \arrow[r] & 0.
\end{tikzcd}$$

The groups on the upper row are simplicial and the groups on the lower row are singular. Rows are known to be exact and the diagram is easily seen to be commutative. By naturality of long exact homology sequence (Lemma 11.9) we obtain the commutative diagram

$$\begin{tikzcd}
H_{m+1}(K^n, K^{n+1}) \arrow[r] \arrow[d, "i_*\cong"] & H_m(K^{n-1}) \arrow[r] \arrow[d, "i_*\cong"] & H_m(K^n) \arrow[r] \arrow[d, "i_*"] & H_m(K^n, K^{n-1}) \arrow[r] \arrow[d, "i_*\cong"] & H_{m-1}(K^{n-1}) \arrow[d, "i_*\cong"] \\
H_{m+1}(|K^n|, |K^{n-1}|) \arrow[r] & H_m(|K^{n-1}|) \arrow[r] & H_m(|K^n|) \arrow[r] & H_m(|K^n|, |K^{n-1}|) \arrow[r] & H_{m-1}(|K^{n-1}|)
\end{tikzcd}$$

where upper row is a part of the long exact simplicial homology sequence of the pair (K^n, K^{n-1}) and the lower row is a part of the long exact singular homology sequence of the pair $(|K^n|, |K^{n-1}|)$. Here all vertical homomorphisms are known to be isomorphism by Lemma 15.2 and inductive assumption (this fact is indicated by the symbol $\cong$ in the diagram next to the mapping), except for the middle one, which is the one we are interested in. The five-lemma 11.14 implies that also the middle mapping is an isomorphism $\iota_*\colon H_m(K^{n-1}) \to H_m(|K^{n-1}|)$. This works for all $m \in \mathbb{Z}$, hence proves the inductive step. The theorem 15.1 is proved in an absolute case.

Relative case now follows easily by a similar trick involving five-lemma. Let (K, L) is a pair of finite Δ-complexes. By the absolute case, which is already proved for all finite complexes, we have that

$$\iota_*\colon H_m(L) \to H_m(|L|) \text{ and}$$

$$\iota_*\colon H_m(K) \to H_m(|K|)$$

are isomorphisms for all $m \in \mathbb{Z}$.

The commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C(L) \arrow[r, "i"] \arrow[d, "\iota"] & C(K) \arrow[r, "j"] \arrow[d, "\iota"] & C(K, L) \arrow[r] \arrow[d, "\iota"] & 0 \\
0 \arrow[r] & C(|L|) \arrow[r, "i"] & C(|K|) \arrow[r, "j"] & C(|K|, |L|) \arrow[r] & 0
\end{tikzcd}$$

of chain complexes and chain mappings with exact rows induces, by Lemma 11.9 the commutative diagram

$$\begin{tikzcd}
H_n(L) \arrow[r] \arrow[d, "i_*"] & H_n(K) \arrow[r] \arrow[d, "i_*"] & H_n(K, L) \arrow[r] \arrow[d] & H_{n-1}(L) \arrow[r] \arrow[d, "i_*"] & H_{n-1}(K) \arrow[d, "i_*"] \\
H_n(|L|) \arrow[r] & H_n(|K|) \arrow[r] & H_n(|K|, |L|) \arrow[r] & H_{n-1}(|L|) \arrow[r] & H_{n-1}(|K|)
\end{tikzcd}$$

with exact rows. Again, by the absolute case, we know that all vertical mappings, except the one in the middle, are isomorphisms. By the five-lemma 11.14 also the middle mapping is an isomorphism. This concludes the proof of the Theorem 15.1 (assuming Lemma 15.2).

It remains to actually prove the Lemma 15.2.

Proof of the Lemma 15.2.

The simplicial homology groups $H_m(K^n, K^{n-1})$ can be easily calculated directly from the definition. Complexes K^n and K^{n-1} have exactly the same (geometrical) simplices in dimensions $m \neq n$. In dimension n complex K^{n-1} do not have simplices. It follows that all groups, except possibly one, of the chain complex $C(K^n, K^{n-1})$ are trivial, so the boundary operators of this complex are zero homomorphisms. It follows that $H_m(K^n, K^{n-1}) = 0$ when $m \neq n$, while $H_n(K^n, K^{n-1})$ is isomorphic to $C_n(K^n, K^{n-1})$ via projection, hence is a free abelian group generated by the set

$$\{\overline{\sigma} \mid \sigma \in K_n/\sim\},$$

in other words by the classes of all geometrical n-simplices of K.

A homomorphism between a free abelian group A and a group B is an isomorphism if and only if it maps some basis of A into a basis of B. By definition of the mapping ι we have that

$$\iota(\sigma) = f_\sigma,$$

where $f_\sigma \colon \Delta_n \to |K|$ is the characteristic mapping of the geometrical simplex σ of K. Hence the Lemma is proved if we can show that $H_m(|K^n|, |K^{n-1}|) = 0$ when $m \neq n$ and $H_n(|K^n|, |K^{n-1}|)$ is a free abelian group with basis

$$\{\overline{f_\sigma} \mid \sigma \in K_n/\sim]\}.$$

For every geometrical n-simplex σ of K we choose an interior point $\mathbf{x}_\sigma \in \operatorname{Int}\sigma$. For instance we can choose x_σ to be "'a barycentre" i.e. the image of the barycentre $\mathbf{b}$ of Δ_n under the characteristic mapping f_σ. Thus we assert, for

$$\mathbf{x}_\sigma = f_\sigma(\mathbf{b}).$$

let $A = |K^{n-1}|$ and let U be the set

$$U = |K| \setminus \{\mathbf{x}_\sigma \mid \sigma \in K_n\}.$$

Then A is closed in $|K^n|$ (as a polyhedron of a subcomplex), U is open (exercise) and $A \subset U$. Hence the inclusion $i \colon (|K^n| \setminus |K^{n-1}|,\ U \setminus |K^{n-1}|) \to$

$(|K^n|, U)$ satisfies the excision property and thus induces isomorphisms in singular homology (Theorem 14.1).

On he other hand the inclusion $j\colon (|K^n|,\ |K^{n-1}|) \to (|K^n|,\ U)$ is a mapping such that both $j = \mathrm{id}\colon |K^n| \to |K^n|$ and $j||K^{n-1}|\colon |K^{n-1}| \to U$ are homotopy equivalences (exercise). Such a mapping always induces isomorphisms in singular homology (Proposition 13.12). Hence there is an isomorphism

$$j_*^{-1} \circ i_*\colon H_n(|K^n| \setminus |K^{n-1}|,\ U \setminus |K^{n-1}|) \to H_n(|K^n|,\ |K^{n-1}|).$$

The space $X = |K^n| \setminus |K^{n-1}|$ is a disjoint union of simplicial interiors $\mathrm{Int}[\sigma]$, where σ goes through all n-dimensional geometric simplices of K. Every interior $\mathrm{Int}\,\sigma$ is open in $|K^n|$. Here one needs to be extremely careful not to make a logical mistake. The claim is not true because they are " interiors "-usually simplicial interiors of simplices in Δ-complexes **are not** open in the polyhedron $|K|$ are are not the same thing as topological interiors of those simplices. However in this special case they are. This is seen as following. Let σ be a geometrical n-simplex of K. Since K^n is a Δ-complex itself, it follows, by Lemma 6.12, that it is enough to show that $g_\tau^{-1}(\mathrm{Int}[\sigma])$ is open in τ for every simplex τ of K^n. Here $g\colon \tau \to |K^n|$ is a standard quotient mapping. If $\dim \tau < n$, then no point of τ can be identified with a point from $\mathrm{Int}\,\sigma$ (simple consequence of Lemma 6.11), so $g_\tau^{-1}(\mathrm{Int}[\sigma])$ is empty. If $\dim \tau = n$, $g_\tau^{-1}(\mathrm{Int}[\sigma])$ is non-empty if and only if $\tau \sim \sigma$ in K, in which case

$$g_\tau^{-1}(\mathrm{Int}[\sigma]) = \mathrm{Int}\,\tau,$$

which is open in τ. The only remaining case $\dim \tau > n$ is not possible, since K^n has no simplices in dimensions bigger than n (here is where the proof would not go through for the whole complex K).

We have shown that the space $X = |K^n| \setminus |K^{n-1}|$ is a disjoint union of simplicial interiors $\mathrm{Int}[\sigma]$, which are all open in X. Since arbitrary unions of open sets are also open,

$$\mathrm{Int}[\sigma] = X \setminus \Big(\bigcup_{[\tau]\neq[\sigma]} \mathrm{Int}[\tau] \Big)$$

is also closed as a complement of an open set. Moreover, by Lemma 6.14 the restriction of the characteristic mapping $f_\sigma\colon \mathrm{Int}\,\Delta_n \to \mathrm{Int}\,\sigma$ is a homeomorphism for all $[\sigma] \in K_n/\sim$. This homeomorphism maps the subset $\mathrm{Int}\,\Delta_n \setminus \{\mathbf{b}\}$ exactly to

$$\mathrm{Int}\,\sigma \setminus \{\mathbf{x}\} = \mathrm{Int}\,\sigma \cap (U \setminus |K^{n-1}|)$$

It follows in particular that every subset $\mathrm{Int}[\sigma]$ is homeomorphic to the open ball B^n, in particular path-connected. Hence the path-components of X are exactly the sets $\mathrm{Int}[\sigma]$. Combining all these results with Lemma 12.3, we see that the mapping

$$\sum((f_{[\sigma]})_*)_m\colon \oplus H_m(\mathrm{Int}\,\Delta_n, \mathrm{Int}\,\Delta_n \setminus \mathbf{b}) \to H_m(|K^n|,\ U)$$

is an isomorphism for all $m \in \mathbb{Z}$.

The inclusion $(\mathrm{Int}\,\Delta_n, \mathrm{Int}\,\Delta_n \setminus \mathbf{b}) \to (\Delta_n, \Delta_n \setminus \mathbf{b}$ is a special case of the mapping of pairs $i\colon (|K^n| \setminus |K^{n-1}|,\ U \setminus |K^{n-1}|) \to (|K^n|, U)$, for the complex $K = K(\Delta_n)$. Hence this inclusion is an excision mapping, that induces an isomorphism in homology, in all dimensions. The diagram

$$\begin{tikzcd}
\oplus H_m(\mathrm{Int}\,\Delta_n, \mathrm{Int}\,\Delta_n \setminus \mathbf{b}) \arrow[dd, "\cong"] \arrow[dr, "\sum(f_\sigma)_*"] & \\
& H_m(|K^n|, U) \\
\oplus H_m(\Delta_n, \Delta_n \setminus \mathbf{b}) \arrow[ur, "\sum(f_{[\sigma]})_*"] &
\end{tikzcd},$$

is easily seen to commute. Hence the mapping

$$\sum((f_{[\sigma]})_*)_m\colon \oplus H_m(\Delta_n, \Delta_n \setminus \mathbf{b}) \to H_m(|K^n|,\ U)$$

is also an isomorphism for all $m \in \mathbb{Z}$.

The inclusion $(\Delta_n, \mathrm{Bd}\,\Delta_n) \to (\Delta_n, \Delta_n \setminus \mathbf{b})$ is a homotopy equivalence (exercise). Hence it induces isomorphisms between homology groups and we can substitute $H_m(\Delta_n, \Delta_n \setminus \mathbf{b})$ with $H_m(\Delta_n, \mathrm{Bd}\,\Delta_n)$ above. In other words the mapping

$$\sum((f_{[\sigma]})_*)\colon \oplus H_m(\Delta_n, \mathrm{Bd}\,\Delta_n) \to H_m(|K^n|,\ U)$$

is an isomorphism for all $m \in \mathbb{Z}$. Since f_σ maps $\mathrm{Bd}\,\Delta_n$ into $|K^{n-1}|$, we can finally substitute the group $H_m(|K^n|,\ U)$ with the group $H_m(|K^n|,\ |K^{n-1}|)$. The mapping remains an isomorphism, since the inclusion $j\colon (|K^n|,\ |K^{n-1}|) \to (|K^n|,\ U)$ is a homotopy equivalence.

We have shown that the mapping

$$\sum((f_{[\sigma]})_*)_m\colon \oplus H_m(\Delta_n, \mathrm{Bd}\,\Delta_n) \to H_m(|K^n|,\ |K^{n-1}|)$$

is an isomorphism for all $m \in \mathbb{Z}$.

Using Lemma 14.11 from the previous section, we obtain from this that $H_m(|K^n|, |K^{n-1}|) = 0$ for $m \neq n$ and $H_n(|K^n|, |K^{n-1}|)$ is a free abelian group with the set of generators $\{[f_\sigma] \mid [\sigma] \in K_n/\sim\}$. This is exactly what needed to be shown and the proof of Lemma 15.2 is now complete. Thus also the main result of this section, the Theorem 15.1, is proved.

The result is quite powerful indeed - in many cases the calculation of simplicial homology is much simpler and much concrete algebraic exercise concerning (finitely generated) free abelian group then the calculation of the singular homology directly from the definition. The result also helps further investigation of the structure of homological groups. As an example let us calculate the concrete generator of the group $H_n(S^n), n > 0$, which we know to be isomorphic to $\mathbb{Z}$.

Example 15.3. *We know that $S^n \cong \operatorname{Bd}\Delta_{n+1}$ and by Corollary 14.12 we know that $\overline{d\,\mathrm{id}}$ is a generator for the group $H_n(\operatorname{Bd}\Delta_{n+1})$. Thus for any homeomorpism $f\colon \operatorname{Bd}\Delta_{n+1} \cong S^n$, the element $f_*([d\,\mathrm{id}])$ is a generator of $H_n(S^n)$. However this element depends on the choice of the homeomorphism f and does not really have a simple and "natural" relation to the structure of S^n, so it might not be useful.*

Let

$$B_+ = \{\mathbf{x} \in S^n | x_{n+1} \geq 0\},$$
$$B_- = \{\mathbf{x} \in S^n | x_{n+1} \leq 0\},$$

and define $i\colon S^n \to S^n$, $i(x_0, \ldots, x_n, x_{n+1}) = (x_0, \ldots, x_n, -x_{n+1})$. Then ι is clearly a homeomorphism which maps B_+ to B_- (and vice versa).
The mapping $\alpha\colon B_+ \to \overline{B}^n$ defined by

$$\alpha(x) = (x_1, \ldots, x_n)$$

is a homeomorphism (exercise). Choose any homeomorphism $\beta\colon \Delta_n \to \overline{B}^n$, then $f = \alpha^{-1} \circ \beta\colon \Delta^n \to B_+ \subset S^n$ is a homeomorphism and can be thought of as an element of $C_n(S^n)$. Likewise $g = i \circ f\colon \Delta_n \to B_-$ is a homeomorphism, that can be identified with an element of $C_n(S^n)$.
It is easy to see that the images of f and g intersect precisely at the "equator"

$$S^{n-1} = B_+ \cap B_- = \{\mathbf{x} \in S^n \mid x_{n+1} = 0.\}$$

which is the image of the boundaries of Δ_n under both mappings. Hence, if we take two n-simplices U and V and identify all their $(n-1)$-faces i.e. identify

the face d^iU with the face d^iV, we obtain a Δ-complex K with $|K| = S^n$. Mappings f and g are then precisely the characteristic mappings of simplices U and V.
We have that $dU = dV \neq 0$ in the simplicial chain group $C_{n-1}(K)$, so $H_n(K) = \operatorname{Ker} d_n$ (since there are no $n+1$-simplices) and an element $kU + lV \in C_n(K)$ is in the kernel of d_n if and only if

$$d(kU + lV) = (k + l)dU = 0 \text{ i.e. if and only if } k + l = 0.$$

It follows that $H_n(K)$ is a free abelian group generated on one element $U - V$. Using the isomorphism $i_*\colon H_n(K) \to H_n(|K|)$ we see immediately that $\overline{f - g}$ is a generator of the group $H_n(S^n)$.

Geometrically this generator can be thought of as "upper hemisphere minus lower hemisphere".

As an application let us calculate the mapping $i_*\colon H_n(S^n) \to H_n(S^n)$ induced by the mapping $i\colon S^n \to S^n$ defined above by

$$i(x_0, \ldots, x_n, x_{n+1}) = (x_0, \ldots, x_n, -x_{n+1}).$$

We have that $i_\sharp(f) = i \circ f = g$ and $i_\sharp(g) = i \circ g = i \circ i \circ f = f$, since $i \circ i = \mathrm{id}$. Hence

$$i_*[f - g] = [g - f] = -[f - g].$$

Thus $i_*(x) = -x$ for all $x \in H_n(S^n)$. Using this fact it is now easy to prove (exercise) that for the antipodal mapping $h\colon S^n \to S^n$, $h(x) = -x$ one has

$$h_*(x) = (-1)^{n+1}x, \ x \in H_n(S^n).$$

Notice that if we would use as a generator an element $f_*(\overline{d\,\mathrm{id}})$, where $f\colon \operatorname{Bd}\Delta_{n+1} \cong S^n$ is a homeomorphism, then it would not provide us with a simple way to calculate the induced homomorphism i_*, since i_* has no obvious relation with $f_*(\overline{d\,\mathrm{id}})$.

Example 15.4. *We can now use the calculations of the simplicial homology of spaces such as the Mobius Band and projective plane from examples 9.6 and 9.7 in the section 9. Since simplicial homology and singular homology groups are isomorphic we obtain for singular groups of the Mobius Band M and projective plane $\mathbb{R}P^2$ results*

$$H_n(M) \cong \begin{cases} \mathbb{Z}, & n = 0, 1 \\ 0, & \textit{otherwise}, \end{cases}$$

$$H_n(\mathbb{R}P^2) \cong \begin{cases} \mathbb{Z}, & n=0, \\ \mathbb{Z}_2, & n=1, \\ 0, & otherwise, \end{cases} .$$

In particular the Mobius band has the same homology groups as the sphere S^1. This is not a coincidence. It can be shown that the Mobius band has the same homotopy type as the sphere S^1 (exercise), so the singular homology groups of the Mobius band can be also calculated using that fact (and homotopy axiom).

Example 15.5. *Consider the Δ-complex K constisting of one 1-simplex $\sigma = \Delta_1 = [0,1]$ with both end points $d^0\sigma = 1$, $d^1\sigma = 0$ identified. The polyhedron $|K|$ of this complex is homeomorphic to S^1. The charachteristic mapping $f_\sigma\colon \Delta_1 \to |K| = S^1$ is actually (up to a homeomorphism) a mapping $\gamma\colon I \to S^1$ defined by*

$$\gamma(t) = (\cos 2\pi t, \sin 2\pi t).$$

The simplicial homology group $H_1(K)$ is easy to calculate. It is free abelian group $\mathbb{Z}[\overline{\sigma}]$ with one generator $\overline{\sigma}$. Since $\iota_*(\overline{\sigma}) = \overline{\gamma}$, it follows by Theorem 15.1, that $\overline{\gamma}$ is a generator of $H_1(S^1)$.

Let $D = \{0 = s_0 < s_1 < \ldots < s_m = 1\}$ be a division of an interval I. For every $i = 1, \ldots, m$ we define $\gamma_i\colon I \to S^1$ by

$$\gamma_i(t) = (cos 2\pi((1-t)s_{i-1} + ts_i), \sin 2\pi((1-t)s_{i-1} + ts_i)).$$

According to the example 9.10 we have that

$$\overline{\gamma} = \overline{\sum_{i=1}^{m} \gamma_i},$$

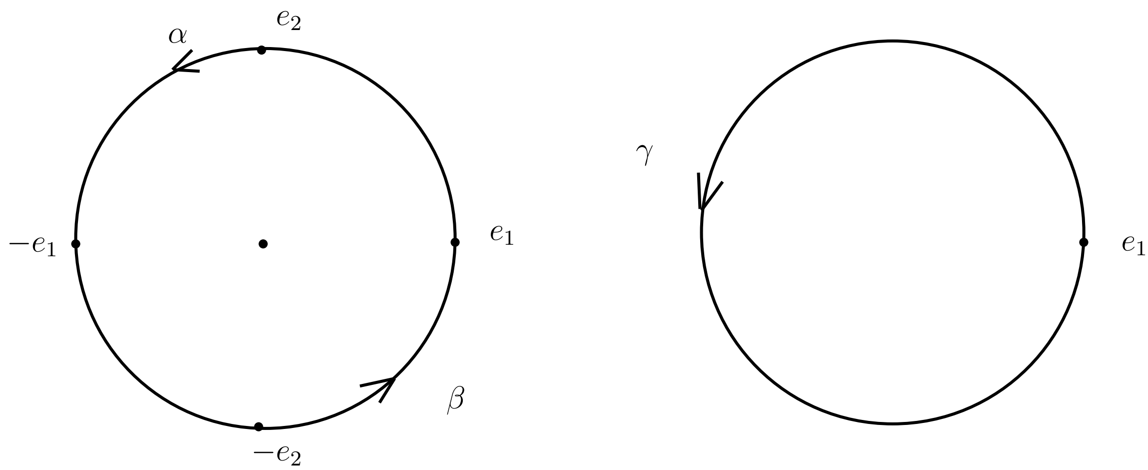
so $\overline{\sum_{i=1}^{m} \gamma_i}$ is also a generator of $H_1(S^1)$ for any choice of the division D. Moreover all these generators are actually the same element.

In particular let $\alpha, \beta\colon I \to S^1$ be defined by

$$\alpha(t) = \cos(\pi t) + i \sin(\pi t),$$

$$\beta(t) = \cos(\pi t + \pi t) + i \sin(\pi + \pi t).$$

Then α represents upper half ark of the circle and β represents lower half ark of the circle.



Choosing $D = \{0, 1/2, 1\}$ we obtain that $\alpha = \gamma_1$, $\beta = \gamma_2$ for this particular division. Hence

$$\overline{\gamma} = \overline{\alpha + \beta}.$$

We will use this equation in the next section in the course of calculating the homology groups of the projective plane.

16 Mayer-Vietoris sequence.

Mayer-Vietoris sequence is an equivalent way to formulate the excision theorem, which in some contexts can be technically more convenient than the statement of the excision property.

Algebraic motivation for the Mayer-Vietoris sequence is quite elementary. Suppose A, B are subgroups of an abelian group G. Then $A + B$ is also a subgroup of G. In general we cannot expect this sum to be the direct sum $A \oplus B$, since the intersection $A \cap B$ might be non-trivial. Nevertheless there exists group homomorphism $q\colon A \oplus B \to A + B$ defined by

$$q(x, y) = x + y,$$

for all $x \in A$, $y \in B$. The mapping j is clearly surjective. An element $(x, y) \in A \oplus B$ belongs to the kernel of q if and only $x + y = 0$ i.e. if and only if $y = -x$, in which case $x, y \in A \cap B$. It follows that if we define a homomorphism $h\colon A \cap B \to A \oplus B$ by $h(x) = (x, -x)$, then $\operatorname{Im} h = \operatorname{Ker} q$. Moreover, it is easy to verify, that the mapping h is injective. Thus there exists a short exact sequence

$$0 \longrightarrow A \cap B \xrightarrow{h} A \oplus B \xrightarrow{q} A + B \longrightarrow 0$$

of abelian groups and homorphism.

Suppose (C, d) is a chain complex and $A, B \subset C$ are subcomplexes. Then $A \cap B$ and $A + B$ are also subcomplexes of C and, by considerations above, for every $n \in \mathbb{Z}$ we have a short exact sequence

$$0 \longrightarrow A_n \cap B_n \xrightarrow{h_n} A_n \oplus B_n \xrightarrow{q_n} A_n + B_n \longrightarrow 0,$$

where $h_n \colon A_n \cap B \to A \oplus B$ and $q_n \colon A \oplus B \to A + B$ are defined as above,

$$h_n(x) = (x, -x),$$

$$q_n(x, y) = x + y.$$

It is easy to check that $h = \{h_n\}$ and $q = \{q_n\}$ are chain mappings (exercise). Hence we obtain a short exact sequence

$$0 \longrightarrow A \cap B \xrightarrow{h} A \oplus B \xrightarrow{q} A + B \longrightarrow 0$$

of chain complexes and chain mappings. As usual, this sequence induces long exact sequence

$$\ldots \longrightarrow H_{n+1}(A + B) \xrightarrow{\Gamma} H_n(A \cap B) \xrightarrow{h_*} H_n(A \oplus B) \xrightarrow{q_*} H_n(A + B) \longrightarrow \ldots$$

in homology (where Γ is the boundary operator). Let $k_1 \colon A \to A \oplus B$ and $k_2 \colon B \to A \oplus B$ be canonical inclusions. By Lemma 12.1

$$k = (k_1)_* + (k_2)_* \colon H_n(A) \oplus H_n(B) \to H_n(A \oplus B)$$

is an isomorphism for all $n \in \mathbb{Z}$. Let $i_1 \colon A \cap B \to A$ and $i_2 \colon A \cap B \to B$ be inclusions. Then

$$h(x) = (x, -x) = (k_1 \circ i_1(x), -k_2 \circ i_2(x)).$$

It follows that (exercise)

$$k^{-1} \circ h_*(z) = ((i_1)_*(z), -(i_2)_*(z)), z \in H_n(A \cap B), \text{ and}$$

$$q_* \circ k(\overline{x}, \overline{y}) = (l_1)_*(\overline{x}) + (l_2)_*(\overline{y}),$$

where $l_1 \colon A \to A + B$, $l_2 \colon B \to A + B$ are inclusions.

Substituting h_* with $k^{-1} \circ h_*$ and q_* with $q_* \circ k$, we obtain long exact sequence

$$\ldots \longrightarrow H_{n+1}(A + B) \xrightarrow{\Gamma} H_n(A \cap B) \xrightarrow{\alpha} H_n(A) \oplus H_n(B) \xrightarrow{\beta} H_n(A + B) \longrightarrow \ldots,$$

which is called the **Mayer-Vietoris sequence** of the pair (A, B). Here $\alpha = ((i_1)_*, -(i_2)_*)$ and $\beta = (l_1)_* + (l_2)_*$.

The boundary operator $\Gamma\colon H_{n+1}(A+B) \to H_n(A \cap B)$ of this sequence is defined as following. Let $c \in A+B$ be a cycle. Then $c = a+b$ for some $a \in A, b \in B$ (not necessarily unique). Moreover,

$$d_n(a) + d_n(b) = d_n(c) = 0.$$

Hence $d_n(a) = -d_n(b) \in A_n \cap B_n$, so by the definition of Γ, we have that

$$\Gamma\overline{c} = \overline{d_n a} = -\overline{d_n(b)}.$$

Example 16.1. *Suppose K is a Δ-complex and L_1, L_2 are subcomplexes of K such that $L_1 \cup L_2 = K$, i.e. every simplex of K is either in L_1 or in L_2 (or both). Then $C(L_1) + C(L_2) = C(K)$, so the Mayer-Vietoris sequence of the pair $(C(L_1), C(L_2))$ is the exact sequence*
(16.2)

$$\ldots \longrightarrow H_{n+1}(K) \longrightarrow H_n(L_1 \cap L_2) \longrightarrow H_n(L_1) \oplus H_n(L_2) \longrightarrow H_n(K) \longrightarrow \ldots,$$

involving simplicial homology groups. Here the simple fact $C(L_1) \cap C(L_2) = C(L_1 \cap L_2)$ was also used. We call the sequence (16.2) the **Mayer-Vietoris sequence of the pair** (L_1, L_2).

In general, when L_1 and L_2 are just subcomplexes of a Δ-complex K, their union $K' = L_1 \cup L_2$ is also a subomplex, so there exists Mayer-Vietoris sequence (16.2) with $K' = L_1 \cup L_2$ in place of K.

Mayer-Vietoris sequence for the singular homology
Suppose X is a topological pair and let $U, V \subset X$. Then $C(U) \cap C(V) = C(U \cap V)$ and there exists Mayer-Vietoris sequence of the pair $C(U), C(C)$, which is the exact sequence
(16.3)

$$\ldots \longrightarrow H_{n+1}(C(U)+C(V)) \longrightarrow H_n(U \cap V) \longrightarrow H_n(U) \oplus H_n(V) \longrightarrow H_n(C(U)+C(V)) \longrightarrow \ldots$$

Usually for the arbitrary subsets U, V the sequence (16.3) is not very useful, since the groups $H_n(C(U) + C(V))$ are rarely interesting. The standard applications of Mayer-Vietoris is a calculation of the groups $H_n(X)$, using the groups $H_n(U)$, $H_n(V)$ and $H_n(U \cap V)$, which are assumed to be known. That is why we are only interested in the special case when we can substitute the group $H_n(C(U)+C(V)$ with $H_n(X)$. This motivates the following definition.

Definition 16.4. *Let U and V be subsets of the topological space X. Let $i\colon C(U) + C(V) \to C(X)$ be the inclusion of a subcomplex $C(U) + C(V)$*

into $C(X)$. We call the triple $(X;U,V)$ **a proper triad** *if the induced homomorphism*

$$i_*\colon H_n(C(U)+C(V)) \to H_n(X)$$

in homology are **isomorphisms** *for all $n \in \mathbb{Z}$.*

Proposition 16.5. *Suppose $(X;U,V)$ is a proper triad. Then there exists exact sequence*

$$\ldots \longrightarrow H_{n+1}(X) \xrightarrow{\Delta} H_n(U\cap V) \xrightarrow{((i_1)_*,-(i_2)_*)} H_n(U)\oplus H_n(V) \xrightarrow{(l_1)_*+(l_2)_*} H_n(X) \xrightarrow{\Delta} \ldots,$$

called the **Mayer-Vietoris sequence** *of the proper triad $(X;U,V)$.*
Here $i_1\colon U\cap V \to U, i_2\colon U\cap V \to V$, $l_1\colon U \to X$ and $l_2\colon V \to X$ are inclusions.

Proof. By considerations above there exist the exact sequence

$$\ldots \longrightarrow H_{n+1}(C(U)+C(V)) \longrightarrow H_n(U\cap V) \longrightarrow H_n(U)\oplus H_n(V) \longrightarrow H_n(C(U)+C(V)) \longrightarrow \ldots$$

Substituting $H_n(C(U)+C(V))$ with an isomorphic group $H_n(X)$ and mappings involved with corresponding mappings, we obtain the claim. $\square$

The theorem 14.6 implies directly the following important result.

Lemma 16.6. *Suppose $U,V \subset X$ are such that*

$$\operatorname{int} U \cup \operatorname{int} V = X.$$

Then $(X;U,V)$ is a proper triad.

Example 16.7. *As an example of the way Mayer-Vietoris sequence can be applied, let us calculate (once more) the groups $H_m(S^n)$, for $n>0$. We only assume the homotopy axiom and other basis properties of homology groups, but not excision.*
Let

$$U = S^n \setminus \{\mathbf{e}_{n+1}\},$$
$$V = S^n \setminus \{-\mathbf{e}_{n+1}\}.$$

Then U and V are open subsets of S^n, so $(S^n;U,V)$ is a proper triad (Lemma 16.6). Hence there exists Mayer-Vietoris sequence

$$\ldots \longrightarrow H_m(U)\oplus H_m(V) \longrightarrow H_m(S^n) \xrightarrow{\Delta} H_{m-1}(U\cap V) \longrightarrow H_{m-1}(U)\oplus H_{m-1}(V) \longrightarrow \ldots,$$

which is exact. Both spaces U and V are homeomorphic to $\mathbb{R}^n$ (Example 3.8), hence contractible, if $n>0$. It follows that, as long as $m \neq 0$, we *have that $H_m(U) = H_m(V) = 0$. Thus, if $m > 1$, Mayer-Vietoris sequence becomes the exact sequence*

$$0 \longrightarrow H_m(S^n) \xrightarrow{\Delta} H_{m-1}(U \cap V) \longrightarrow 0,$$

thus $\Delta\colon H_m(S^n) \to H_{m-1}(U \cap V)$ is an isomorphism of groups (by exactness). But, the space $U \cap V$ is homeomorphic to $\mathbb{R}^n$ minus a point, which has the same homotopy type as S^{n-1}. Thus $H_{m-1}(U \cap V) \cong H_{m-1}(S^{n-1})$, and we obtain that

$$H_m(S^n) \cong H_{m-1}(S^{n-1}).$$

Now we can "slide down" arriving, by induction, to the case $n = 1$. The only technical problem is that $H_0(U)$ and $H_0(V)$ are not trivial and the proof above works only as long as $m > 1$. For $m = 1$ the sequence is of the form

$$0 \longrightarrow H_1(S^n) \xrightarrow{\Delta} H_0(U \cap V) \longrightarrow H_0(U) \oplus H_0(V) \longrightarrow H_0(S^n) \longrightarrow 0,$$

where $H_0(U) \cong \mathbb{Z}$, $H_0(V) \cong \mathbb{Z}$ and the mapping $H_0(U) \oplus H_0(V) \to H_0(S^n)$ is induced by inclusions $U \to S^n, V \to S^n$. By exactness $\Delta\colon H_1(S^n) \to H_0(U \cap V)$ is an injection, whose image is (by exactness) the same as the kernel of $H_0(U \cap V) \to H_0(U) \oplus H_0(V)$. Hence we need to investigate this mapping first. There are two cases - if $n > 1$, then space $U \cap V$, as well as U and V are path-connected, so all groups $H_0(U \cap V), H_0(U), H_0(V)$ are isomorphic to $\mathbb{Z}$ and the mapping $H_0(U \cap V) \to H_0(U) \oplus H_0(V)$ essentially looks like the mapping $\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, $n \to (n, -n)$. This mapping is injective, so its kernel is trivial. It follows that the image of Δ is trivial, so, since Δ is injective, we must have $H_1(S^n) = 0$.
The case $n = 1$ is different. When $n = 1$, we have that $U \cap V = S^1 \setminus \{\mathbf{e}_2, -\mathbf{e}_2\}$ has two path components. It is easy to check (exercise) that the mapping $H_0(U \cap V) \to H_0(U) \oplus H_0(V)$ is now essentially the mapping $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ given by the formula $(n, m) \to (n+m, -(n+m))$. The kernel of this mapping is a subgroup

$$\{(n, -n) \mid n \in \mathbb{Z}\}$$

of $\mathbb{Z} \times \mathbb{Z}$. This subgroup is a free abelian group with one generator $(1, -1)$, hence isomorphic to $\mathbb{Z}$.

We have shown that $H_1(S^1)$ is isomorphic to $\mathbb{Z}$ and $H_1(S^n) = 0$ for $n > 1$. To calculate $H_m(S^n)$ by induction, using the isomorphism $H_m(S^n) \cong H_{m-1}(S^{n-1})$ ($m > 1$), that we have shown to exist above, we need to calculate $H_m(S^1)$ for all $m \in \mathbb{Z}$ first. This claim will become a initial step in inductive *proof of the statement*

$$H_m(S^n) = \begin{cases} \mathbb{Z}, & m = n, 0 \\ 0, & m \neq n, 0 \end{cases},$$

where $n \geq 1$. We will calculate the initial case $n = 1$, leaving the inductive step as an exercise to the reader.

For $m = 0$ we know that $H_0(S^1) \cong \mathbb{Z}$, since S^1 is path-connected. For $m = 1$ we have already calculated $H_m(S^1) \cong \mathbb{Z}$ Also for $m < 0$ the claim is clear. We have to show that $H_m(S^1) = 0$ for $m > 1$. In this case we have the portion of the Mayer-Vietoris sequence (see above)

$$0 \longrightarrow H_m(S^1) \xrightarrow{\Delta} H_{m-1}(U \cap V) \longrightarrow 0,$$

where $U \cap V$ have the homotopy type of S^0. Since $m > 1$, we have that $H_{m-1}(U \cap V) = 0$, and we are done. Notice that we also use the fact that $H_m(S^0) = 0$ for $m > 0$, which can be proved, as earlier, directly.

The lack of symmetry in cases $m = 1$ and $m > 1$ in the previous example has lead to complicated technical calculations, where the case $m = 1$ had to be treated separately. Reader might have a suspicion that the the calculation would be easier if one would use reduced groups instead of absolute groups. That is true, but in order to perform such a proof, we need the version of the Mayer-Vietoris sequence for the reduced groups.

Proposition 16.8. *Suppose $(X; U, V)$ is a proper triad, such that $U \cap V \neq \emptyset$. Then there exists exact sequence*

$$\ldots \longrightarrow \widetilde{H}_{n+1}(X) \xrightarrow{\Delta} \widetilde{H}_n(U \cap V) \xrightarrow{(i_1)_*, -(i_2)_*} \widetilde{H}_n(U) \oplus \widetilde{H}_n(V) \xrightarrow{(k_1)_* + (k_2)_*} \widetilde{H}_n(X) \xrightarrow{\Delta} \ldots,$$

called the **reduced Mayer-Vietoris sequence** *of the proper triad $(X; U, V)$. Here $i_1 \colon U \cap V \to U, i_2 \colon U \cap V \to V$, $k_1 \colon U \to X$ and $k_2 \colon V \to X$ are inclusions.*

The proof of the previous proposition is a standard application of homological algebra (namely Lemma 11.11) and is left to the reader as an exercise.

Example 16.9. *Let's see how much more easier the calculation of the homology groups of the sphere S^n using Mayer-Vietoris would be, if we would use reduced groups instead. Let $n > 0$. We use the same subsets*

$$U = S^n \setminus \{\mathbf{e}_{n+1}\},$$

$$V = S^n \setminus \{-\mathbf{e}_{n+1}\}$$

as before. The triad $(S^n; U, V)$ is still proper (Lemma 16.6). Moreover $U \cap V$ is non-empty when $n > 0$ (notice - when $n = 0$ it is empty). Hence there exists reduced Mayer-Vietoris sequence

$$\ldots \longrightarrow \widetilde{H}_m(U) \oplus \widetilde{H}_m(V) \longrightarrow \widetilde{H}_m(S^n) \xrightarrow{\Delta} \widetilde{H}_{m-1}(U \cap V) \longrightarrow \widetilde{H}_{m-1}(U) \oplus \widetilde{H}_{m-1}(V) \longrightarrow \ldots,$$

which is exact. Both spaces U and V are homeomorphic to $\mathbb{R}^n$ (Example 3.8), hence contractible, so the reduced groups $\widetilde{H}_m(U), \widetilde{H}_m(V)$ are trivial **for all** *$m \in \mathbb{Z}$ (this is symmetry that absolute groups lack). Thus, the reduced Mayer-Vietoris sequence becomes the exact sequence*

$$0 \longrightarrow \widetilde{H}_m(S^n) \xrightarrow{\Delta} \widetilde{H}_{m-1}(U \cap V) \longrightarrow 0,$$

thus $\Delta \colon \widetilde{H}_m(S^n) \to \widetilde{H}_{m-1}(U \cap V)$ is an isomorphism of groups (by exactness). Since $U \cap V$ have the same homotopy type as S^{n-1}, there is an isomorphism $\widetilde{H}_m(S^n) \to \widetilde{H}_{m-1}(S^{n-1})$ for all $m \in \mathbb{Z}$. Since reduced homology groups of S^0 can be calculated directly from definition, the general case can be deduced from this by induction.

Compare this to the calculation in the previous example and learn an important lesson - reduced groups are much more convenient for the concrete calculations.

Mayer-Vietoris sequence is easily seen to be natural with respect to the mappings of proper triads. To be precise let $(X; U, V)$ and $(Y; Z, W)$ be proper triads and suppose $f \colon X \to Y$ is a continuous mapping with $f(U) \subset Z$, $f(V) \subset W$. We also denote this as $f \colon (X; U, V) \to (Y; Z, W)$. Then the diagram

$$\begin{array}{ccccccccc}
H_{n+1}(X) & \longrightarrow & H_n(U \cap V) & \longrightarrow & H_n(U) \oplus H_n(V) & \longrightarrow & H_n(X) & \longrightarrow & H_{n-1}(U \cap V) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f|_*} & & \downarrow{\scriptstyle f|_* \oplus f|_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f|_*} \\
H_{n+1}(Y) & \longrightarrow & H_n(Z \cap W) & \longrightarrow & H_n(Z) \oplus H_n(W) & \longrightarrow & H_n(Y) & \longrightarrow & H_{n-1}(Z \cap W)
\end{array}$$

where both rows are portions of the corresponding Mayer-Vietoris sequences, commutes, for all $n \in \mathbb{Z}$. The simple verification of this claim is left to the reader.

Example 16.10. *As a more complicated example let us calculate the homology groups of the projective plane $\mathbb{R}P^n$ for all $n \geq 1$.*

There exist two models for $\mathbb{R}P^n$, that we shall need in this example. The traditional way to define projective space is as a quotient space $X_n = S^n/\sim$ of the sphere S^n defined by the equivalence relation $\sim$, which is generated by relations $x \sim -x$ for all $x \in S^n$.

There is also another model for $\mathbb{R}P^n$. Define an equivalence relation $\sim'$ on the ball $\overline{B^n}$ generated by the relations $x \sim' -x$ for all $x \in S^{n-1}$. Notice that the identifications happen only on the boundary and B^n remain "untouched". We define $Y_n = \overline{B}^n/\sim'$.
Define the mapping $p\colon S^n \to \overline{B}^n/\sim' = Y_n$ by

$$p(x_1,\dots,x_{n+1}) = \begin{cases} \overline{(x_1,\dots,x_n)} \text{ if } x_{n+1} \geq 0 \\ \overline{(-x_1,\dots,-x_n)} \text{ if } x_{n+1} \leq 0 \end{cases} .$$

Then p is a well-defined mapping. Moreover it is continuous and closed, in particular a quotient mapping (exercise). The relation $\sim_p$ defined by p on S^n is exactly the relation $\sim$ (exercise). Hence, by Corollary 6.2) p induces homeomorphism $\overline{p}\colon X_n = S^n/\sim \to \overline{B}^n/\sim' = Y_n$. Hence we can identify $X = \mathbb{R}P^n = Y$. Notice that under this identification the mapping $p\colon S^n \to \mathbb{R}P^n = Y_n$ is exactly a projection quotient space.

Since we consider $\mathbb{R}^n$ a subset of $\mathbb{R}^{n+1}$, we can also consider S^{n-1} as a subset of S^n. This means that we identify S^{n-1} with a subset ("equator")

$$S^n \cap \mathbb{R}^n = \{(x_1,\dots,x_n,0) \in S^n\}$$

of S^n, which is homeomophic to S^{n-1}. Consider the restriction $q = p|S^{n-1}\colon S^{n-1} \to Y_n$ of p. Then $q(\mathbf{x}) = q(\mathbf{y})$ if and only if $\mathbf{y} = \pm\mathbf{x}$. Also, $q\colon S^{n-1} \to q(S^{n-1})$ is a closed mapping. This is a consequence of the general topological fact, that asserts that whenever $f\colon X \to Y$ is a closed mapping and $A \subset Y$, then the restriction mapping $f|f^{-1}(A)\colon f^{-1}A \to A$ is also a closed mapping. Choosing $f = p$, $A = q(S^{n-1})$ gives a claim.

It follows, by Corollary 6.2, that q induces a homeomorphism $\overline{q}\colon \mathbb{R}P^{n-1} \to q(S^{n-1}) \subset \mathbb{R}P^n$. Here we are using the model $X_{n-1} = S^{n-1}/\sim$ for $\mathbb{R}P^{n-1}$. Thus we can think of $\mathbb{R}P^{n-1}$ as a subset of $\mathbb{R}P^n$ in a natural way. In the model X_n this is an image of the equator S^{n-1} with respect to the projection mapping $S^n \to \mathbb{R}P^n$. In the model Y_n this is an image of the boundary S^{n-1} with respect to the projection mapping $\overline{B}^n \to \mathbb{R}P^n$.

Let $q\colon \overline{B}^n \to \overline{B}^n/\sim' = \mathbb{R}P^n$ to be a quotient mapping. Then q is a closed mapping (exercise), so the restriction $q|B^n\colon B^n \to q(B^n)$ is of the form $q|q^{-1}U\colon q^{-1}U \to U$, where $U = q(B^n)$, hence also closed, by the general *topological result above. Also, this restriction is bijective. Hence it is a homeomorphism. The subset U is open in the quotient space $\mathbb{R}P^n$, since $q^{-1}(U) = B^n$ is open in $\overline{B}^n$ and q is a quotient mapping. We also denote $V = q(\overline{B}^n \setminus \{0\})$. Since $q^{-1}V = \overline{B}^n \setminus \{0\}$ is open in $\overline{B}^n$, V is also open in $\mathbb{R}P^n$. Clearly $U \cup V = \mathbb{R}P^n$. Hence the triad $(\mathbb{R}P^n; U, V)$ is proper, so there exists a reduced Mayer-Vietoris exact sequence*

$$\ldots \longrightarrow \widetilde{H}_{m+1}(\mathbb{R}P^n) \longrightarrow \widetilde{H}_m(U \cap V) \longrightarrow \widetilde{H}_m(U) \oplus \widetilde{H}_m(V) \longrightarrow \widetilde{H}_m(X) \longrightarrow \ldots$$

The subspace $\mathbb{R}P^{n-1}$ is a subspace of V. Moreover the inclusion $\mathbb{R}P^{n-1} \to V$ is a homotopy equivalence, exactly for the same reason that $S^{n-1} \hookrightarrow \overline{B}^n \setminus \{0\}$ is a homotopy equivalence (exercise).
Hence we can substitute the group $\widetilde{H}_m(V)$ above with the isomorphic group $\widetilde{H}_m(\mathbb{R}P^{n-1})$. Also U is contractible (since it is homeomorphic to B^n), so its reduced homology groups are all trivial. What about the subset $U \cap V$? Since U is homeomorphic to B^n, $U \cap V$ is homeomorphic to the punctured open ball $B^n \setminus \{0\}$ in a natural way. Clearly it has the same homotopy type as its subpace

$$S' = \{x \in B^n \mid |x| = 1/2\},$$

which is homeomorphic to S^{n-1}. More precisely the inclusion $S' \to U \cap V$ is a homotopy equivalence, hence induces an isomorphism in reduced homology groups. If we substitute $\widetilde{H}_m(U \cap V)$ with isomorphic group $\widetilde{H}_m(S^{n-1})$, then the inclusion $U \cap V \to V$ will be substituted with a quotient projection $p\colon S^{n-1} \to \mathbb{R}P^{n-1} = X_{n-1}$ (check it!). Hence we obtain an exact sequence
(16.11)

$$\ldots \longrightarrow \widetilde{H}_{m+1}(\mathbb{R}P^n) \xrightarrow{\Delta} \widetilde{H}_m(S^{n-1}) \xrightarrow{p_*} \widetilde{H}_m(\mathbb{R}P^{n-1}) \xrightarrow{i_*} \widetilde{H}_m(\mathbb{R}P^n) \xrightarrow{\Delta} \widetilde{H}_{m-1}(S^{n-1}) \longrightarrow \ldots$$

where p_* is induced by the projection $p\colon S^{n-1} \to \mathbb{R}P^{n-1} = X_{n-1}$ and i_* is induced by the inclusion $i\colon \mathbb{R}P^{n-1} \to \mathbb{R}P^n$.

Since $\widetilde{H}_{n-1}(S^{n-1})$ is the only non-trivial reduced homology group of S^{n-1}, the exact sequence 16.11 implies the following facts
1) The homomorphism $i_*\colon \widetilde{H}_m(\mathbb{R}P^{n-1}) \to \widetilde{H}_m(\mathbb{R}P^n)$ induced by inclusion is an isomorphism for $m \neq n, n-1$.
2) There is an exact sequence

$$0 \longrightarrow \widetilde{H}_n(\mathbb{R}P^{n-1}) \xrightarrow{i_*} \widetilde{H}_n(\mathbb{R}P^n) \xrightarrow{\Delta} \widetilde{H}_{n-1}(S^{n-1}) \xrightarrow{p_*} \widetilde{H}_{n-1}(\mathbb{R}P^{n-1}) \xrightarrow{i_*} \widetilde{H}_{n-1}(\mathbb{R}P^n) \longrightarrow 0.$$

So, if we want to calculate homology groups of $\mathbb{R}P^n$ by induction on n, in the inductive step we need not only to know the homology groups of the previous *case* $\mathbb{R}P^{n-1}$ *but also a homomorphism* $p_*\colon \widetilde{H}_{n-1}(S^{n-1}) \to \widetilde{H}_{n-1}(\mathbb{R}P^{n-1})$.

Let us start with the initial step $n = 1$. *The second model for* $\mathbb{R}P^1$ *shows immediately that* $\mathbb{R}P^1 = [-1,1]/\{1,-1\}$ *is a closed interval with end points identified, hence essentially* S^1. *Moreover under this identification* $p\colon S^1 \to \mathbb{R}P^1 \cong S^1$ *looks like the mapping* $S^1 \to S^1$ *that wraps the upper ark of* S^1 *around* S^1 *one time, and then does the same for the lower ark. More precisely this means (exercise) that*

$$p(\cos 2\pi t, \sin 2\pi t) = (\cos 4\pi t, \sin 4\pi t)$$

for all $t \in I$.

Recall the mappings $\alpha, \beta, \gamma\colon I \to S^1$ *from Example 15.5, defined by*

$$\alpha(t) = \cos(\pi t) + i\sin(\pi t),$$

$$\beta(t) = \cos(\pi t + \pi t) + i\sin(\pi + \pi t).$$

$$\gamma(t) = (\cos 2\pi t, \sin 2\pi t).$$

According to the Example 15.5

$$\overline{\gamma} = \overline{\alpha + \beta}$$

is a generator for $H_1(S^1) \cong \mathbb{Z}$.
By definition we have that

$$p_\sharp(\alpha) = p \circ \alpha = \gamma = p \circ \beta = p_\sharp(\beta), \text{ so}$$

$$p_*(\overline{\gamma}) = p_*(\overline{\alpha + \beta}) = \overline{\gamma + \gamma} = 2\overline{\gamma}.$$

Since $\overline{\gamma}$ *is a generator of the free group* $H_1(S^1) \cong \mathbb{Z}$, *we have that*

$$p_*(x) = 2x \text{ for all } x \in H_1(S^1).$$

In other words $p_*\colon H_1(S^1) \to H_1(S^1)$ *looks like the homomorphism* $\mathbb{Z} \to \mathbb{Z}$, $n \mapsto 2n$.

Since $\mathbb{R}P^1$ *is homeomorphic to* S^1, *we have that* $\widetilde{H}_m(\mathbb{R}P^1) = 0$ *for* $m > 1$. *Since* $\widetilde{H}_m(\mathbb{R}P^n)$ *is isomorphic to* $\widetilde{H}_m(\mathbb{R}P^{n-1})$ *for* $m > n$, *this implies immediately by induction on* m *that* $\widetilde{H}_m(\mathbb{R}P^n) = 0$ *for* $m > n$. *Hence the exact sequence above becomes the exact sequence*

$$0 \longrightarrow \widetilde{H}_n(\mathbb{R}P^n) \xrightarrow{\Delta} \widetilde{H}_{n-1}(S^{n-1}) \xrightarrow{p_*} \widetilde{H}_{n-1}(\mathbb{R}P^{n-1}) \xrightarrow{i_*} \widetilde{H}_{n-1}(\mathbb{R}P^n) \longrightarrow 0.$$

By exactness this sequence implies that $\widetilde{H}_n(\mathbb{R}P^n)$ is a subgroup $\operatorname{Ker} p_*$ *of* $\widetilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$*, so, in particular it is always either trivial, or isomorphic to $\mathbb{Z}$, since all subgroups of $\mathbb{Z}$ are either trivial or of the form $k\mathbb{Z}$, $k > 0$. This is true for all $n > 0$. Hence $p_* \colon \widetilde{H}_{n-1}(S^{n-1}) \to \widetilde{H}_{n-1}(\mathbb{R}P^{n-1})$ is always essentially a homomorphism $\mathbb{Z} \to \mathbb{Z}$. Such a homomorphism is always of the form $z \mapsto kz$ for some fixed $k \in \mathbb{Z}$. It follows that p_* is either a zero homomorphism or injective. In case it is injective, we have that $\widetilde{H}_n(\mathbb{R}P^n) =$* $\operatorname{Ker} p_* = 0$*. In case it is a zero homomorphism, $\widetilde{H}_n(\mathbb{R}P^n) = \operatorname{Ker} p_* = \widetilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$. Hence in particular we obtain the following conclusion.*
Suppose $\widetilde{H}_n(\mathbb{R}P^n) = 0$. Then in the next dimension we will have exact sequence

$$0 \longrightarrow \widetilde{H}_{n+1}(\mathbb{R}P^{n+1}) \xrightarrow{\Delta} \widetilde{H}_n(S^n) \longrightarrow 0 \longrightarrow \widetilde{H}_n(\mathbb{R}P^{n+1}) \longrightarrow 0.$$

By exactness this sequence implies that $\widetilde{H}_{n+1}(\mathbb{R}P^{n+1}) \cong \widetilde{H}_n(S^n) \cong \mathbb{Z}$ and $\widetilde{H}_n(\mathbb{R}P^{n+1}) = 0$.

Let us continue by induction and consider the next case $n = 2$. Since we have seen that $p_* \colon \widetilde{H}_1(S^1) \to \widetilde{H}_1(\mathbb{R}P^1)$ is an injection $x \mapsto 2x$, it follows that $\widetilde{H}_2(\mathbb{R}P^2) = \operatorname{Ker} p_* = 0$ and $\widetilde{H}_1(\mathbb{R}P^2) \cong \widetilde{H}_1(\mathbb{R}P^1)/\operatorname{Im} p_* = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$.
From the considerations from the proceeding paragraph it now follows that $\widetilde{H}_3(\mathbb{R}P^3) \cong \mathbb{Z}$, $\widetilde{H}_2(\mathbb{R}P^3) = 0$ and $\widetilde{H}_1(\mathbb{R}P^3) = \widetilde{H}_1(\mathbb{R}P^2) = \mathbb{Z}_2$.
For the next case $n = 4$ we need to know the mapping $p_* \colon \mathbb{Z} = \widetilde{H}_3(S^3) \to \widetilde{H}_3(\mathbb{R}P^3) = \mathbb{Z}$, so let us investigate this matter in general.

To make use of the naturality of Mayer-Vietoris sequence it makes sense to define proper triad $(S^n; Z, W)$ such that $p \colon S^n \to \mathbb{R}P^n$ will be mapping of triads $p \colon (S^n; Z, W) \to (\mathbb{R}P^n; U, V)$. Miming the definition of U and V and keeping in mind the definition of p we define

$$Z = Z_+ \cup Z -_= \{\mathbf{x} \in S^n \mid x_{n+1} > 0\} \cup \{x \in S^n \mid x_{n+1} < 0\},$$

$$W = S^n \setminus \{\mathbf{e}_{n+1}, -\mathbf{e}_{n+1}\}.$$

Then $(S^n; Z, W)$ is a proper triad, since Z and W are both open and cover S^n. Moreover $p \colon (S^n; Z, W) \to (\mathbb{R}P_n, U, V)$, so we have a commutative diagram

$$\begin{tikzcd}
\widetilde{H}_{m+1}(S^n) \arrow[r, "\Delta"] \arrow[d, "p_*"] & \widetilde{H}_m(Z \cap W) \arrow[r, "i_*"] \arrow[d, "p|_*"] & \widetilde{H}_m(Z) \oplus \widetilde{H}_m(W) \arrow[r, "j_*"] \arrow[d, "p|_* \oplus p|_*"] & \widetilde{H}_m(S^n) \arrow[r, "\Delta"] \arrow[d, "p_*"] & \widetilde{H}_{m-1}(Z \cap W) \arrow[d, "p|_*"] \\
\widetilde{H}_{m+1}(\mathbb{R}P^n) \arrow[r, "\Delta"] & \widetilde{H}_m(U \cap V) \arrow[r, "i_*"] & \widetilde{H}_m(U) \oplus \widetilde{H}_m(V) \arrow[r, "j_*"] & \widetilde{H}_m(\mathbb{R}P^n) \arrow[r, "\Delta"] & \widetilde{H}_{m-1}(U \cap V)
\end{tikzcd}$$

with exact rows. Now let us simplify the upper row in the same way we have already simplified the lower row. The subspace $Z \cup W$ has the same homotopy type as its subspace $S_+ \cup S_-$, where

$$S_+ = \{x \in S^n \mid x_{n+1} = \sqrt{3}/4\},$$

$$S_- = \{x \in S^n \mid x_{n+1} = -\sqrt{3}/4\},$$

via the inclusion $S_+ \cup S_- \to Z \cap W$. The reason we have chosen the weird looking number $\sqrt{3}/4$ above is that then p maps $S_+ \cup S_-$ onto $S' = \{\mathbf{x} \in B^n \mid |x| = 1/2\} \subset \mathbb{R}P_n$. Since both S_+ and S_- are homeomorphic to S^{n-1} in an obvious way, we can write the restriction of p to $S_+ \cup S_-$ as a mapping $S^{n-1} \sqcup S^{n-1} \to S^{n-1}$ ($\sqcup$ denotes the disjoint topological union) defined by the identity mapping id on the first copy of S^{n-1} (corresponding to S_+) and by the antipodal mapping $h\colon S^{n-1} \to S^{n-1}$, $h(x) = -x$ on the second summond corresponding to S^{n-1}. In particular $p|_*\colon \widetilde{H}_m(Z \cup W) \to \widetilde{H}_m(U \cap V)$ in the diagram above becomes the homomorphism $\mathrm{id} + h_*\colon \widetilde{H}_m(S^{n-1}) \oplus \widetilde{H}(S^{n-1}) \to \widetilde{H}_m(S^{n-1})$.

The subspace Z is a disjoint union of two contractible spaces, so $\widetilde{H}_m(Z) = 0$ for $m \in \mathbb{Z}$. The subspace W has the same homotopy type as its subspace $S^{n-1} = \{x \in S^n \mid x_{n+1} = 0\}$. Substituting spaces with their subspaces with the same homotopy type, we see that $p\colon W \to V$ becomes $p\colon S^{n-1} \to \mathbb{R}P^{n-1}$ and $i\colon Z \cap W \to W$ becomes $\mathrm{id} \sqcup \mathrm{id}\colon S^{n-1} \sqcup S^{n-1} \to S^{n-1}$. Summarizing all these information we obtain the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{H}_n(S^n) \arrow[r] \arrow[d, "p_*"] & \widetilde{H}_{n-1}(S^{n-1}) \oplus \widetilde{H}_{n-1}(S^{n-1}) \arrow[r, "\mathrm{id}+\mathrm{id}"] \arrow[d, "\mathrm{id} \oplus h_*"] & \widetilde{H}_{n-1}(S^{n-1}) \arrow[r, "j_*"] \arrow[d, "p_*"] & \widetilde{H}_{n-1}(S^n) \arrow[r] \arrow[d, "p_*"] & 0 \\
0 \arrow[r] & \widetilde{H}_n(\mathbb{R}P^n) \arrow[r, "\Delta"] & \widetilde{H}_{n-1}(S^{n-1}) \arrow[r, "p_*"] & \widetilde{H}_{n-1}(\mathbb{R}P^{n-1}) \arrow[r] & \widetilde{H}_{n-1}(\mathbb{R}P^n) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. The interesting case is the case when $p_*\colon H_{n-1}(S^{n-1}) \to H_{n-1}(\mathbb{R}P^{n-1})$ is a zero mapping, since in the other case p_* is injective, so we can say for certain that $H_n(\mathbb{R}P^n) = 0$. Thus suppose that $p_*\colon H_{n-1}(S^{n-1}) \to H_{n-1}(\mathbb{R}P^{n-1})$ is a zero mapping. Then, by exactness of the diagram above, we have the commutative diagram

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "\Delta\cong"] \arrow[d, "p_*"] & G \arrow[d, "(id_*+h_*)|"] \\
H_n(\mathbb{R}P^n) \arrow[r, "\Delta\cong"] & H_{n-1}(S^{n-1}),
\end{tikzcd}$$

where G is the kernel of the mapping

$$(\mathrm{id} + \mathrm{id})\colon H_{n-1}(S^{n-1}) \oplus H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$$

and both boundary operators Δ are isomorphisms .
Now $\mathrm{id} + \mathrm{id}\colon H_{n-1}(S^{n-1}) \oplus H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$ *is essentially the mapping $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}, (n, m) \mapsto n + m$, so*

$$\mathrm{Ker}(\mathrm{id} + \mathrm{id}) = \{(x, -x) \in H_{n-1}(S^{n-1}) \oplus H_{n-1}(S^{n-1})) \mid x \in H_{n-1}(S^{n-1})\}.$$

It can be shown, using example 15.3, that $h_*(x) = (-1)^n x$ for all $x \in H_{n-1}(S^{n-1})$. Hence

$$(\mathrm{id}_* + h_*)(x, -x) = x + (-1)^n(-x) = x + (-1)^{n+1}(x) = \begin{cases} 0, & \textit{if } n \textit{ is even,} \\ 2x, & \textit{if } n \textit{ is odd.} \end{cases} .$$

By the commutativity of the diagram above, the mapping $p_*\colon H_n(S^n) \to H_n(\mathbb{R}P^n)$ has the same description i.e. as a mapping $\mathbb{Z} \to \mathbb{Z}$ looks like the zero mapping, if n is even, and the mapping $x \mapsto 2x$, if n is odd, at least when suitable generators are used.

Let us prove by induction that whenever $H_n(\mathbb{R}P^n) \cong \mathbb{Z}$ only the second case actually occurs. To be precise we claim that
(1) if n is even, then $H_n(\mathbb{R}P^n) = 0$ and $H_{n-1}(\mathbb{R}P^n) \cong \mathbb{Z}_2$,
(2) if n is odd, then $H_n(\mathbb{R}P^n) \cong \mathbb{Z}, \tilde{H}_{n-1}(\mathbb{R}P^n) = 0$ and $p_*\colon H_n(S^n) \to H_n(\mathbb{R}P^n)$ is essentially the mapping $n \mapsto 2n$.

For $n = 1$ we have already shown the claim to be true. Suppose n is odd and the claim is true for $n-1$, which is even. Then, by inductive assumption $H_{n-1}(\mathbb{R}P^{n-1}) = 0$ and the considerations above apply, showing that the claim is true also for n.
If n is even and the claim is true for $n-1$, which is then odd, then $p_*\colon H_n(S^{n-1}) \to H_n(\mathbb{R}P^{n-1})$is the injection with image $2\mathbb{Z} \subset \mathbb{Z}$, so the exact sequence above shows that $H_n(\mathbb{R}P^n) = 0$ and $H_{n-1}(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$.

Putting together all the information about the homology groups of $\mathbb{R}P^n$ we have managed to obtain, we get the following.

$$H_m(\mathbb{R}P_n) = \begin{cases} \mathbb{Z}, & \textit{for } m = 0, \\ \mathbb{Z}_2, & \textit{for } 0 < m < n \textit{ if } m \textit{ is odd ,} \\ \mathbb{Z}, & \textit{for } m = n \textit{ if } n \textit{ is odd ,} \\ 0, & \textit{otherwise.} \end{cases}$$

All the examples of proper triads so far we of the form $(X; U, V)$, where $\operatorname{int} U \cup \operatorname{int} V = X$. There are also other important classes of proper triads.

Proposition 16.12. *Suppose K is a Δ-complex and L_1 and L_2 are subcomplexes of K such that $K = L_1 \cup L_2$. Then $(|K|; |L_1|, |L_2|)$ is a proper triad.*

Proof. By Example 16.1 there exists long exact sequence

$$\ldots \longrightarrow H_{n+1}(K) \longrightarrow H_n(L_1 \cap L_2) \longrightarrow H_n(L_1) \oplus H_n(L_2) \longrightarrow H_n(K) \longrightarrow \ldots$$

involving simplicial homology groups. On the other hand in any case, there always exists long exact sequence

$$\ldots \longrightarrow H_{n+1}(C) \longrightarrow H_n(|L_1| \cap |L_2|) \longrightarrow H_n(|L_1|) \oplus H_n(|L_2|) \longrightarrow H_n(C) \longrightarrow \ldots,$$

where $C = C(|L_1|) + C(|L_2|)$, see (16.3). There exist canonical chain mappings $\iota\colon C(L_1 \cap L_2) \to C(|L_1| \cap |L_2|)$, (note that $|L_1 \cap L_2| = |L_1| \cap |L_2|$), $\iota\colon C(L_1) \to C(|L_1|)$, $\iota\colon C(L_2) \to C(|L_2|)$ and $\iota\colon C(K) \to C(|K|)$. All these mappings induce isomorphisms in homology, by Theorem 15.1. On the other hand, since $K = L_1 \cup L_2$, it is easy to see that $\iota\colon C(K) \to C(|K|)$ maps $C(K)$ into a subcomplex $C(|L_1|) + C(|L_2|)$ of $C(K)$. Hence there exists commutative diagram

$$\begin{tikzcd}
H_n(L_1 \cap L_2) \arrow[r] \arrow[d, "\iota"] & H_n(L_1) \oplus H_n(L_2) \arrow[r] \arrow[d, "\iota"] & H_n(K) \arrow[r] \arrow[d, "\iota_*"] & H_{n-1}(L_1 \cap L_2) \arrow[r] \arrow[d, "\iota"] & H_{n-1}(L_1) \oplus H_{n-1}(L_2) \arrow[d, "\iota"] \\
H_n(|L_1| \cap |L_2|) \arrow[r] & H_n(|L_1|) \oplus H_n(|L_2|) \arrow[r] & H_n(C) \arrow[r] & H_{n-1}(|L_1| \cap |L_2|) \arrow[r] & H_{n-1}(|L_1|) \oplus H_{n-1}(|L_2|)
\end{tikzcd}$$

with exact rows. Here again C stands for a complex $C(|L_1|) + C(|L_2|)$. In this diagram all vertical mappings are isomorphisms, except possibly for the middle mapping $\iota_*\colon H_n(K) \to H_n(C(|L_1|) + C(|L_2|))$. Five Lemma 11.14 implies that this mapping is also an isomorphism, for all $n \in \mathbb{Z}$. Now, in the commutative diagram

$$\begin{tikzcd}
H_n(K) \arrow[rr, "\iota_*"] \arrow[dr, "\iota_*"'] & & H_n(C(|L_1|) + C(|L_2|)) \arrow[dl, "i_*"] \\
& H_n(|K|) &
\end{tikzcd}$$

both mappings ι_* are isomorphisms. Hence also $i_*\colon H_n(C(|L_1|)+C(|L_2|)) \to H_n(|K|)$ is an isomorphism. This means precisely that $(|K|; |L_1|, |L_2|)$ is a proper triad. $\square$

Example 16.13. *Let K be the Δ-complex generated by two n-simplices σ_1, σ_2, with all their corresponding faces $d^i\sigma_1$ and $d^i\sigma_2$ (as well as the corresponding lower dimensional faces) identified. Let L_j be the subcomplex of K generated by σ_j, $j = 1, 2$. Then $K = L_1 \cup L_2$, so the previous result implies that $(|K|; |L_1|, |L_2|)$ is a proper triad. It is easy to see that $(|K|; |L_1|, |L_2|)$ is homeomorphic to the triad (S^n, S^n_+, S^n_-), where*

$$S^n_+ = \{\mathbf{x} \in S^n \mid x_{n+1} \geq 0\},$$

$$S^n_- = \{\mathbf{x} \in S^n \mid x_{n+1} \leq 0\}$$

are the upper and lower hemispheres. Both subpaces S^n_+ and S^n_- are homeomorphic to $\overline{B}^n$, in particular contractible. Moreover, $S^n_+ \cap S^n_- = S^{n-1}$. Hence the reduced Mayer-Vietoris sequence

$$\widetilde{H}_m(S^n_+) \oplus \widetilde{H}_m(S^n_-) \longrightarrow \widetilde{H}_m(S^n) \xrightarrow{\Delta} \widetilde{H}_{m-1}(S^{n-1}) \longrightarrow \widetilde{H}_{m-1}(S^n_+) \oplus \widetilde{H}_{m-1}(S^n_-) \longrightarrow \dots$$

is the exact sequence

$$0 \longrightarrow \widetilde{H}_m(S^n) \xrightarrow{\Delta} \widetilde{H}_{m-1}(S^{n-1}) \longrightarrow 0,$$

which implies directly the existence of the homomorphism $\widetilde{H}_m(S^n) \cong \widetilde{H}_{m-1}(S^{n-1})$, for all $m \in \mathbb{Z}$. So far, this is probably the simplest and shortest way to calculate homology groups of the sphere by induction, that we have encountered.

17 Some classical applications

In this section we will apply the machinery of singular homology theory to prove some important classical topological results, such as the Brouwer's fixed point theorem, invariance of domain and Jordan-Brouwer separation theorem.

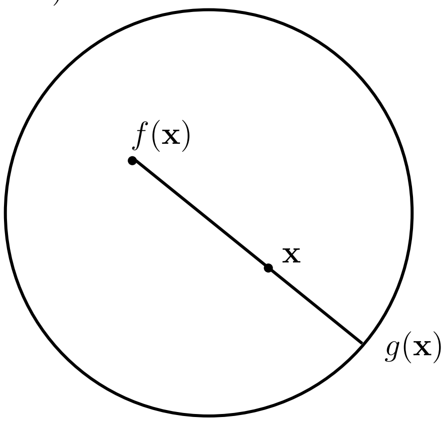
Theorem 17.1. The Brouwer's fixed point theorem
Suppose $C \subset V$ is a compact and convex non-empty subset of the finite-dimensional vector space V. Then every continuous mapping $f : C \to C$ has a fixed point.

Proof. Since C is homeomorphic to $\overline{B}^n$ for some $n \in \mathbb{N}$ (Theorem 3.20), it is enough to consider the case $C = \overline{B}^n$.

The claim is proved by counter-assumption. Suppose $f\colon \overline{B}^n \to \overline{B}^n$ is a continuous mapping without a fixed point. Then $f(\mathbf{x}) \neq \mathbf{x}$ for all $\mathbf{x} \in \overline{B}^n$. For every $\mathbf{x} \in \overline{B}$ consider the open half-line

$$L_{\mathbf{x}} = \{(1-t)f(\mathbf{x}) + t\mathbf{x} \mid t > 0\}$$

starting at $f(\mathbf{x})$ (not including it however) and going through $\mathbf{x}$ (see the picture).



For every $\mathbf{x} \in \overline{B}^n$ let $g(\mathbf{x})$ be the unique point of S^{n-1} that belongs to $L_{\mathbf{x}}$. This defines a mapping $g\colon \overline{B}^n \to S^{n-1}$. The exact construction of the formula for g and verification of the facts that g is well-defined and continuous is left as an exercise.

By construction it follows that $g(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in S^{n-1}$. Hence g is a retraction of $\overline{B}^n$ onto S^{n-1}. However this is a contradiction with Corollary 14.5, which says that such retraction cannot exist. $\square$

There is a generalization of the Brouwer's fixed point theorem, that says that any mapping $f\colon C \to C$, where C is a contractible compact polyhedra, has a fixed point, but the proof is much more difficult and requires the development of some further machinery.

To prove invariance of domain and Jordan's separation theorem we need some technical results. Next Lemma says that singular homology has "compact carriers".

Lemma 17.2. *Suppose X is a topological space.*

(1) Suppose $x \in H_n(X)$, for some $n \in \mathbb{Z}$. Then there exists compact subset C of X such that $x = j_*(y)$ for some $y \in H_n(C)$, where $j\colon C \to X$ is inclusion.

(2) *Suppose C is a compact subset of X, $j\colon C \to X$ is inclusion. Suppose that $y \in H_n(C)$ is such that $j_*(y) = 0 \in H_n(X)$. Then, there exists compact $D \subset X$ such that $C \subset D$ and $j'_*(y) = 0$, where $j'\colon C \to D$ is inclusion.*

Similar claims are true for reduced homology groups.

Proof. Both properties follow from the fact that singular homology is defined in terms of continuous images of simplices, which are compact.

(1) Suppose $x \in H_n(X)$. Then $x = \overline{\alpha}$ for some cycle $\alpha \in Z_n(X) \subset C_n(X)$. In particular

$$\alpha = \sum_{i=1}^{m} n_i f_i$$

for some $n_1, \ldots, n_m \in \mathbb{Z}$ and some singular n-simplices $f_1, \ldots, f_m\colon \Delta_n \to X$. Since Δ_n is compact, continuous images of compact spaces are compact and finite union of compact spaces is compact, the subset

$$C = \bigcup_{i=1}^{m} f_i(\Delta_n) \subset X$$

is compact. Moreover $\alpha \in C_n(C)$. The boundary in the subgroup $C_n(C)$ is calculated in the same way as in $C_n(X)$, so $\alpha \in Z_n(C)$, hence the homology class $y = \overline{\alpha} \in H_n(C)$ exists. Clearly $i_*(y) = x$.
In reduced case, if $\alpha \in \tilde{C}(X)$, then also $\alpha \in \tilde{C}(C)$.

(2) This is similar to (1) and left to the reader as an exercise. $\square$

Lemma 17.3. *Suppose $B \subset S^n$ is homeomorphic to $\overline{B}^k$ for some $0 \leq k \leq n$. Then $\tilde{H}_m(S^n \setminus B) = 0$ for all $m \in \mathbb{N}$.*

Proof. The proof goes by induction on k. If $k = 0$, then B is a point and the claim is clear, since then $S^n \setminus B$ is homeomorphic to $\mathbb{R}^n$ (Example 3.8), hence contractible.

Let $k > 0$ and suppose that the claim is true for $k-1$. Since the k-dimensional cube I^k is homeomorphic to $\overline{B}^k$, there exists a homeomorphism $f\colon I^k \to B$. Let

$$C_1 = f(I^{k-1} \times [0, 1/2]),$$
$$C_2 = f(I^{k-1} \times [1/2, 1]).$$

Since C_1 and C_2 are both compact, $\{S^n \setminus C_1, S^n \setminus C_2\}$ is an open covering of the set
$$(S^n \setminus C_1) \cup (S^n \setminus C_2) = S^n \setminus (C_1 \cap C_2) = S^n \setminus C.$$
Here $C = C_1 \cap C_2 = f(I^{k-1} \times \{1/2\})$ is homeomorphic to I^{k-1}. Also
$$(S^n \setminus C_1) \cap (S^n \setminus C_2) = S^n \setminus (C_1 \cup C_2) = S^n \setminus B.$$
By Proposition 16.6 $(S^n \setminus C; S^n \setminus C_1, S^n \setminus C_2)$ is a proper triad. Hence there exists reduced Mayer-Vietoris sequence
$$\tilde{H}_{m+1}(S^n \setminus C) = 0 \longrightarrow \tilde{H}_m(S^n \setminus B) \xrightarrow{(i_*,-i_*)} \widetilde{H}_m(S^n \setminus C_1) \oplus \tilde{H}_m(S^n \setminus C_2) \longrightarrow \widetilde{H}_n(S^n \setminus C) = 0$$
where $\tilde{H}_{m+1}(S^n \setminus C) = 0 = \widetilde{H}_n(S^n \setminus C) = 0$ by inductive assumption, since C is homeomorphic to $\overline{B}^{k-1}$. By exactness we see that the homomorphism
$$(i_*, -i_*) \colon \widetilde{H}_m(S^n \setminus B) \to \widetilde{H}_m(S^n \setminus C_1) \oplus \tilde{H}_m(S^n \setminus C_2)$$
is an isomorphism, for all $m \in \mathbb{Z}$.

Fix $m \in \mathbb{Z}$. We want to prove that $\widetilde{H}_m(S^n \setminus B) = 0$. Let us make a counter-assumption - there exists $x \in \widetilde{H}_m(S^n \setminus B)$ such that $x \neq 0$. Then, since $(i_*, -i_*) \colon \widetilde{H}_m(S^n \setminus B) \to \widetilde{H}_m(S^n \setminus C_1) \oplus \tilde{H}_m(S^n \setminus C_2)$ is an isomorphism, either $i_*(x) \neq 0 \in \widetilde{H}_m(S^n \setminus C_1)$ or $i_*(x) \neq 0 \in \widetilde{H}_m(S^n \setminus C_2)$. Choose $B_1 = C_1$ in the first case and $B_1 = C_2$ in the second case.
Since B_1 satisfies the same assumptions as B (as well as the counter-assumption $\widetilde{H}_m(S^n \setminus B_1) \neq 0$), we may repeat the same reasoning applied to B_1, to obtain B_2. Continuing like that (by induction) we obtain an infinite nested sequence
$$B = B_0 \supset B_1 \supset B_2 \supset \ldots \supset B_l \supset B_{l+1} \supset \ldots$$
such that $i^l_*(x) \neq 0$ for all inclusions $i^l \colon (S^n \setminus B) \to (S^n \setminus B_l)$ and $\bigcap_{i=0}^{\infty} B_i = B_\infty$ is homeomorphic to I^{k-1}, hence satisfies the inductive assumption. In other words $\widetilde{H}_m(S^n \setminus B_\infty) = 0$.

Let $i \colon (S^n \setminus B) \to (S^n \setminus B_\infty)$ be the inclusion, then $i_*(x) = 0$. By Lemma 17.2, there exists compact $C \subset S^n \setminus B$ such that $x = j_*(y)$ for some $y \in \tilde{H}_n(C)$, where $j \colon C \to S^n \setminus B$ is an inclusion. Let $j' \colon C \to S^n \setminus B_\infty$ is the inclusion. Then $j'_*(y) = 0 \in \widetilde{H}_m(S^n \setminus B_\infty) = 0$. By Lemma 17.2 there exists compact $D \subset S^n \setminus B_\infty$ such that $C \subset D$ and $j''_*(y) = 0$ for the inclusion $j'' \colon C \to D$.

The collection
$$\{S^n \setminus B_l \mid l \in \mathbb{N}\}$$
is an open covering of the set $S^n \setminus B_\infty$, hence also of D. Since D is compact there exists $q \in \mathbb{N}$ such that

$$D \subset \bigcup_{i=0}^{q} S^n \setminus B_l = S^n \setminus B_q.$$

The diagram

$$\begin{tikzcd}
\widetilde{H}_m(C) \arrow[r, "j''_*"] \arrow[d, "j"] & \widetilde{H}_m(D) \arrow[d, "k_*"] \\
\widetilde{H}_m(S^n \setminus B) \arrow[r, "i^q_*"] & \widetilde{H}_m(S^n \setminus B_q),
\end{tikzcd}$$

where $k\colon D \to S^n \backslash B^q$ and $i^q\colon S^n \backslash B \to S^n \backslash B^q$ are inclusions, is commutative. This implies that

$$i^q_*(x) = i^q_*(j(y)) = k_*(j''_*(y)) = k_*(0) = 0.$$

This however contradicts the construction of B^q. Hence counter-assumption was false, so the claim is true also for k. □

Lemma 17.4. *Suppose $B \subset S^n$ is homeomorphic to S^k for $0 \le k \le n-1$. Then*
$$\widetilde{H}_m(S^n \setminus B) = \begin{cases} 0, & \textit{for } m \neq n-k-1, \\ \mathbb{Z}, & \textit{for } m = n-k-1. \end{cases}$$

Proof. Lemma is proved by induction on k.

If $k = 0$, then $B = \{a, b\}$ is space consisting of two isolated points. Since S^n minus a point is homeomorphic to $\mathbb{R}^n$ (example 3.8), S^n minus two points is homeomorphic to $\mathbb{R}^n$ minus a point, i.e. essentially homeomorphic to the space $\mathbb{R}^n \setminus \{\mathbf{0}\}$. This space, on the other hand, has a homotopy type of the sphere S^{n-1}. Hence in this case

$$\widetilde{H}_m(S^n \setminus B) = \begin{cases} 0, & \text{for } m \neq n-1, \\ \mathbb{Z}, & \text{for } m = n-1, \end{cases}$$

which is exactly the claim for $k = 0$.

Suppose claim is true for $k-1 \ge 0$ and let us show it for k. Let $f\colon S^k \to B$ be a homeomorphism. Denote $C_1 = f(S_+)$ and $C_2 = f(S_-)$, where S_+, S_- are upper and lower hemisphere of S^k, as usual. Since hemispheres are compact, the sets C_1, C_2 are compact, hence closed in Hausdorff space S^n. It follows that their complements are open. Hence $\{S^n \setminus C_1, S^n \setminus C_2\}$ is an open covering of the space $S^n \setminus (C_1 \cap C_2) = S^n \setminus C$, where $C = C_1 \cap C_2$ is homeomorphic to the sphere S^{k-1}. Also

$$(S^n \setminus C_1) \cap (S^n \setminus C_2) = S^n \setminus B.$$

Since S_+ and S_- are both homeomorphic to the closed ball $\overline{B}^k$, the same is true for subsets C_1 and C_2. Hence, by the previous lemma both spaces $S^n \setminus C_1$ and $S^n \setminus C_2$ have trivial reduced groups in all dimensions. The Mayer-Vietoris sequence of the proper triad $(S^n \setminus C; S^n \setminus C_1, S^n \setminus C_2)$, implies, in the usual way, that $\Delta\colon \widetilde{H}_{m+1}(S^n \setminus C) \to \widetilde{H}_m(S^n \setminus B)$ is an isomorphism for all $m \in \mathbb{N}$. Since, by inductive assumption,

$$\widetilde{H}_m(S^n \setminus C) = \begin{cases} 0, & \text{for } m \neq n-k, \\ \mathbb{Z}, & \text{for } m = n-k, \end{cases}$$

it follows that

$$\widetilde{H}_m(S^n \setminus B) = \begin{cases} 0, & \text{for } m \neq n-k-1, \\ \mathbb{Z}, & \text{for } m = n-k-1. \end{cases}$$

□

Theorem 17.5. (Jordan-Brouwer separation theorem).
Suppose B is a subset of S^n homeomorphic to S^{n-1}. Then the space $S^n \setminus B$ has exactly two path components U and V, which are both open in $S^n \setminus B$. Moreover

$$\partial U = B = \partial V,$$

where the topological boundary of both subsets U and V is taken in S^n.

Proof. The case $k = n-1$ in the previous lemma shows that $\widetilde{H}_0(S^n \setminus B) = \mathbb{Z}$. Hence

$$H_0(S^n \setminus B) \cong \widetilde{H}_0(S^n \setminus B) \oplus \mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}.$$

By the lemma 12.5 this implies that $S^n \setminus B$ has exactly two path components, U and V.

Let $\mathbf{x} \in B$ be arbitrary. By Example 3.8 S^n minus a point $\mathbf{x}$ is homeomorphic to $\mathbb{R}^n$. Since $S^n \setminus B$ is a subset of $S^n \setminus \{\mathbf{x}\}$, it follows that $S^n \setminus B$ is homeomorphic to a subset of $\mathbb{R}^n$. Moreover, since B is compact (being homeomorphic to S^{n-1}), it is closed, so $S^n \setminus B$ is open in S^n, in particular it is open in $S^n \setminus \{\mathbf{x}\}$. It follows that $S^n \setminus B$ is homeomorphic to an open subset of $\mathbb{R}^n$. So, in order to prove that it is components U and V are open in it, it is enough to prove in general, that path components of the open subsets of $\mathbb{R}^n$ are open. The proof of this simple (well-known) topological fact is left to the reader as an exercise.

Thus both U and V are open in $S^n \setminus B$. Since $S^n \setminus B$ is open in S^n, this implies that both U and V are open in S^n. It follows that $U \subset S^n \setminus V$, where $S^n \setminus V$ is closed in S^n. Since the closure (taken in S^n) $\overline{U}$ is the smallest closed subset of S^n, which contains U, we have that

$$\overline{U} \subset S^n \setminus V = U \cup B.$$

Since U is open, we also have that

$$\overline{U} = \operatorname{int} U \cup \partial U = U \cup \partial U,$$

where the last union is **disjoint**. Combining these results, we obtain that the boundary ∂U must be a subset of B, $\partial U \subset B$. By the same reasoning $\partial V \subset B$.

It remains to prove that $B \subset \partial U \cap \partial V$. Since B does not intersect U or V, it is enough to show that $B \subset \overline{U} \cap \overline{V}$. Suppose $\mathbf{x} \in B$ and let W be a neighbourhood of $\mathbf{x} \in S^n$. Then $B \cap W$ is a neighbourhood of $\mathbf{x} \in B$. Since B is homeomorphic to S^{n-1}, there exists a neighbourhood A of $\mathbf{x}$ in B such that $A \subset B \cap W$ and $B \setminus A$ is homeomorphic to $\overline{B}^{n-1}$ (why? make sure you understand why!).

By lemma 17.3

$$\widetilde{H}_0(S^n \setminus (B \setminus A)) = 0,$$

hence, by Lemma 12.5, the set

$$S^n \setminus (B \setminus A) = U \cup V \cup A$$

is path-connected. Hence there exist a path $p\colon I \to S^n \setminus (B \setminus A)$ from a point $p(0) \in U$ to the point $p(1) \in V$ (both chosen arbitrary). Since $U \cup V$ is not path-connected and U, V are exactly its path-components, the path p cannot stay in the set $U \cup V$. Hence p must intersect the set A. In fact let

$$t_0 = \sup\{t \in I \mid p(t) \in U\}.$$

Then clearly $p(t_0) \in \overline{U}$ and $p(t_0) \notin U \cup V$, since U and V are both open. Thus $p(t_0) \in A \cap \overline{U}$. Similarly we see that $A \cap \overline{V} \neq \emptyset$. Since $A \subset W$, it follows that in particular W intersects both $\overline{U}$ and $\overline{V}$. Since W is open, this implies that W must actually intersect both U and V (exercise). Since W is an arbitrary neighbourhood of $\mathbf{x} \in B$, it follows that $\mathbf{x} \in \overline{U} \cap \overline{V}$. Since $\mathbf{x} \in B$ is arbitrary, it follows that

$$B \subset \overline{U} \cap \overline{V}.$$

The theorem is proved. $\square$

If $n \geq 2$ the Jordan-Brouwer theorem separation also holds in $\mathbb{R}^n$ (what about the case $n = 1$?). More precisely, if a subset S of $\mathbb{R}^n$ is homeomorphic to S^{n-1}, $n \geq 2$, then $\mathbb{R}^n \setminus S$ has exactly two path components and S is a boundary of both of them. This easily follows from the proved version of the theorem for S^n, since $\mathbb{R}^n$ is homeomorphic to S^n minus a point. Detailed proof is left to the reader as exercise.

Theorem 17.6. Invariance of Domain *Suppose U, V are homeomorphic subsets of S^n. If U is open in S^n, also V is.*

Proof. Let $h\colon U \to V$ be a homeomorphism. It is enough to prove that for every $\mathbf{x} \in U$ there is a neighourhood $Z \subset U$ of $\mathbf{x}$ such that $h(Z)$ is open in S^n.

Let W be small enough neighbourhood of $\mathbf{x}$ in S^n contained in U, such that W is homeomorphic to $\overline{B}^n$ and its boundary A is homeomorphic to S^{n-1}. The existence of such W follows easily from the fact that S^n minus a point is homeomorphic to $\mathbb{R}^n$. Now, W and $h(W)$ are both homeomorphic to $\overline{B}^n$, while A and $h(A)$ are both homeomorphic to S^{n-1}. Also

$$S^n = S^n \setminus h(W) \cup h(A) \cup h(W \setminus A)$$

and this union is disjoint. The set $S^n \setminus h(W)$ is path-connected by Lemma 17.3. The set $h(W \setminus A)$ is also path connected, since $W \setminus A$ is path-connected (it is homeomorphic to B^n). On the other hand by the Jordan-Brouwer separation theorem the set

$$S^n \setminus h(A) = S^n \setminus h(W) \cup h(W \setminus A)$$

has exactly two path-connected components. It follows that these components must be exactly the sets $S^n \setminus h(W)$ and $h(W \setminus A)$. Just like in the proof of Jordan-Brouwer theorem, we notice that the path components of the open subset $S^n \setminus h(A)$ of S^n must be open. In particular $h(W \setminus A)$ is open. Hence $Z = W \setminus A$ is an open neighbourhood of $\mathbf{x}$, whose image is also open. $\square$

Invariance of Domain Theorem is also true for subsets of $\mathbb{R}^n$, just like Jordan-Brouwer separation theorem, for all $n \in \mathbb{N}$. This time the proof of this fact is an elementary corollary of Theorem 17.6 - $\mathbb{R}^n$ is (homeomorphic to the) open subset of S^n, so its subsets are open if and only their open in S^n.

Let

$$\mathbb{H}_n = \{x \in \mathbb{R}^n | x_n \geq 0\}.$$

A non-empty topological space M is called a **topological n-manifold (with boundary)** if
1) M is Hausdorff and
2) M is locally homeomorphic to $\mathbb{R}^n$ or $\mathbb{H}_n$. Precisely these means that every point x of M has a neighbourhood U which is homeomoprhic to an open subset of $\mathbb{R}^n$ or open subset of $\mathbb{H}_n$.
This definition is a bit redundant since every open subset of $\mathbb{R}^n$ is clearly homeomorphic to an open subset of $\mathbb{H}_n$.

In the definition of the manifold it is also customary to include the additional demand that M is second countable or paracompact, but we don't need such technical requirements in this context.

Suppose M is a manifold. Any homeomorphism $f: U \to f(U) \subset M$, where U is an open subset of $\mathbb{R}^n$ or $\mathbb{H}_n$ and $f(U)$ is open in M is called **a chart** in M.
A point $x \in M$ is called **a boundary point** if there is a chart $f: U \to f(U)$ such that U is an open subset of $\mathbb{H}_n$ and $x = f(y)$ for some

$$y \in \{z \in \mathbb{H}^n \mid z_n = 0\}.$$

The set of all boundary points of M is denoted by ∂M.
The point $x \in M$ is called **an interior point** if there is a chart $f: U \to f(U)$ such that U is an open subset of $\mathbb{R}^n$. The set of all interior points is denoted by $\operatorname{int} M$.
If $\partial M = \emptyset$ (which means that all possible charts of M are defined on the open subsets of $\mathbb{R}^n$), we say that M is a manifold **without boundary**.

Once more one needs to be careful with the concepts of interior and boundary. The interior and the boundary of a manifold defined above are not the same as the topological interior and boundary. They are absolute, while their topological cousins are relative. The usage of such terminology is, admittedly, abuse of notation, but it so common in the literature, that we must commit it to.

Using invariance of domain and other information available to us it is easy to prove the following results concerning manifolds. The proofs are left to the reader.

Lemma 17.7. *Suppose M is an n-manifold. Then the sets ∂M and $\operatorname{int} M$ are disjoint. The interior $\operatorname{int} M$ is open in M and itself is an n-manifold without boundary. The boundary ∂M is closed in M and is an $(n-1)$-manifold without boundary. An n-manifold cannot be homeomorphic to an m-manifold if $m \neq n$.*

The last assertion of the previous lemma says that the concept of the "dimension" of a manifold is well-defined.

Lemma 17.8. *Suppose M is an m-manifold, N is an n-manifold.*
1) If $m > n$ there are no continuous injections $M \to N$.
2) If $m = n$ and M has no boundary, then any continuous injection $f\colon M \to N$ is an open embedding, i.e. a homeomorphism to the image $f(M)$, which is open in N (and is a subset of $\operatorname{int} N$).

In particular we obtain the following result, mentioned before. In the literature this result is also often referred to as "the invariance of domain".

Corollary 17.9. *Suppose $f\colon U \to \mathbb{R}^n$ is an injective continuous mapping, where U is an open subset of $\mathbb{R}^n$. Then f is an embedding and $f(U)$ is open in $\mathbb{R}^n$.*

Corollary 17.10. *Suppose M is a compact n-manifold without boundary and N is a connected n-manifold. If $f\colon M \to N$ is a continuous injection, then it it a surjective homeomorphism.*

Proof. By the previous lemma $f(M)$ is open in N. On the other hand $f(M)$ is compact, since M is compact, so $f(M)$ is also closed in N. Since N is connected $f(M) = N$. □

Example 17.11. *Examples of n-manifolds without boundary include $\mathbb{R}^n$ (and all open subsets of $\mathbb{R}^n$), sphere S^n, projective plane $\mathbb{R}P^n$, torus and Klein bottle, which are 2-manifolds. Mobius band is a 2-manifold with boundary. Closed disk $\overline{B}^n$ is an n-manifold with boundary.*

Example 17.12. *The claim 2) in Lemma 17.8 is not true if M has boundary. For example consider the mapping $f\colon [0,1[\to S^1$, $f(t) = e^{2\pi t i}$. Then f is a continuous bijection between two 1-manifolds but it is not homeomorphism.*

Also if $m < n$ there might be continuous injection from m-manifold M to an n-manifold N, which is not embedding, even if M has no boundary. For example let $f\colon]0,1[\to \mathbb{R}^2$ be a mapping defined as in the picture below. Then f is not embedding and the image $f]0,1[$ is not even a manifold

Corollary 17.10 shows that if M is compact n-manifold without boundary and N is a connected n-manifold, which is not compact, then M cannot be embedded in N. This implies, that, for example, the sphere S^n cannot be embedded in $\mathbb{R}^n$.
There is also more precise result known as **the Borsuk-Ulam theorem**, which says that for any mapping $f\colon S^n \to \mathbb{R}^n$ there is $\mathbf{x} \in S^n$ such that $f(\mathbf{x}) = f(-\mathbf{x})$, but the proof of this theorem is too difficult for us at this point.

18 The degree of a mapping

Suppose $n \geq 1$ and $f\colon S^n \to S^n$ is a continuous mapping. Then the induced mapping $f_*\colon H_n(S^n) \to H_n(S^n)$ "looks like" the homomorphism $\mathbb{Z} \to \mathbb{Z}$.
Precisely $H_n(S^n)$ is a free group on one generator. There are precisely two choices for this generator - if a is a generator, then $-a$ is the other possibility. Fix a certain generator a. Then there exists unique $m \in \mathbb{Z}$ such that $f_*(a) = ma$. Moreover n does not depend on the choice of the generator, since then $f_*(-a) = -f_*(a) = -ma = m(-a)$. Also in this case for every $x \in H_n(S^n)$ we have
$$f_*(x) = mx$$
and this property characterizes m uniquely.

Definition 18.1. *Suppose $f\colon S^n \to S^n$ is a continuous mapping ($n \geq 1$). The unique $m \in \mathbb{Z}$ for which*
$$f_*(x) = mx, x \in H_n(S^n)$$
is called **the degree** *of the mapping f and denoted* $\deg f$.

Proposition 18.2. *1)* $\deg \mathrm{id} = 1$.
2) $\deg(g \circ f) = \deg g \cdot \deg f$.
3) If $f \simeq g$ are homotopic, then $\deg f_* = \deg g_*$.
4) If f is not surjective, then $\deg f_* = 0$.

5) If f is a homotopy equivalence, then $\deg f_* = \pm 1$.
6) Suppose $h\colon S^n \to S^n$ is antipodal mapping $h(x) = -x$. Then $\deg h = (-1)^{n+1}$

Proof. Collection of exercises. $\square$

Lemma 18.3. *Suppose $f, g \in S^n \to S^n$ are such that $f(x) \neq -g(x)$ for all $x \in S^n$. Then f and g are homotopic. In particular* $\deg f = \deg g$.

Proof. The assumption of the Lemma implies that the interval

$$\{(1-t)f(x) + tg(x) \mid t \in [0,1]\} \subset \mathbb{R}^n$$

from $f(x)$ to $g(x)$ does not contain 0 (check!). Hence the mapping $H\colon S^n \times I \to S^n$ defined by

$$H(x,t) = \frac{(1-t)f(x) + tg(x)}{|(1-t)f(x) + tg(x)|}$$

is a well-defined and continuous homotopy from f to g. $\square$

Corollary 18.4. *Suppose $f\colon S^n \to S^n$, where n is even. Then there is exists $\mathbf{x} \in S^n$ such that either $f(\mathbf{x}) = \mathbf{x}$ or $f(\mathbf{x}) = -\mathbf{x}$.*

Proof. Let us assume that such a point does not exist. Then for the mappings id and h (antipodal mapping), we have that $f(x) \neq -\mathrm{id}(x)$ and $f(x) \neq -h(x)$. By the previous lemma $\deg f_* = \deg \mathrm{id} = 1$ and at the same time $\deg f_* = \deg h_* = (-1)^{n+1} = -1$, since n is even. This is a contradiction. $\square$

Recall that the standard inner product $\cdot$ on the Euclidean space $\mathbb{R}^n$ is a mapping $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i.$$

A continuous mapping $f\colon S^n \to \mathbb{R}^{n+1}$ is called **a tangent vector field** if $\mathbf{x}, \cdot f(\mathbf{x}) = 0$ for all $\mathbf{x} \in S^n$. Geometrically this could be interpreted as the assignment of a tangent vector at every point of S^n, i.e. an "arrow" that is perpendicular to the surface of S^n at this point, in continuous fashion. This explains the name of the following proposition. It asserts that a ball of even dimension (for example S^2) cannot be "comped" without "bold" spots.

Theorem 18.5. Hairy Ball's theorem.
Suppose $f\colon S^n \to \mathbb{R}^{n+1}$ is a tangent vector field, where n is even. Then there exists a point $\mathbf{x} \in S^n$ such that $f(\mathbf{x}) = 0$. In other words there is no non-zero vector fields on S^n.

Proof. Suppose $f\colon S^n \to R^{n+1}$ is a non-zero vector field. Then f defines a mapping $g\colon S^n \to S^n$ by $g(\mathbf{x}) = f(\mathbf{x})/|f(\mathbf{x})|$. By the properties of the inner product it follows that

$$\mathbf{x} \cdot g(\mathbf{x}) = 0$$

for all $\mathbf{x} \in S^n$. By the previous corollary there is a point $\mathbf{x} \in S^n$ such that either $g(\mathbf{x}) = \mathbf{x}$ or $g(\mathbf{x}) = -\mathbf{x}$. In both cases we obtain

$$\mathbf{x} \cdot \mathbf{x} = \mathbf{x} \cdot (\pm g(\mathbf{x})) = \pm(\mathbf{x}, g(\mathbf{x})) = 0.$$

By the properties of inner product this implies that $\mathbf{x} = 0$. This is impossible, since $\mathbf{x} \in S^n$. $\square$

Both corollary 18.4 and the Hairy Ball's theorem are not true for odd values of $n \in \mathbb{Z}$ (exercise).

We conclude this section by showing that for every $m, n \in \mathbb{Z}$ there is a mapping $f\colon S^n \to S^n$ with $\deg f = m$.

First we need to recall the concept of complex number. A complex number is an element of the plane $\mathbb{R}^2$ i.e. a pair $z = (x, y)$ of real numbers. Complex numbers can be added together and multiplied via algebraic operations $+, \cdot$,

$$(x, y) + (x', y') = (x + x', y + y'),$$
$$(x, y) \cdot (x', y') = (xx' - yy', xy' + x'y).$$

When we think of the plane $\mathbb{R}^2$ equipped with those operations, we call it the set of complex numbers and denote $\mathbb{C}$.
The set $\mathbb{C}$ equipped with addition $+$ is an abelian group. The zero element is the pair $(0, 0)$ and an opposite of a complex number (x, y) is a complex number $(-x, -y)$. The set $\mathbb{C}$ equipped with a multiplication $\cdot$ is **not** an abelian group, since origin $(0, 0)$ do not have an inverse with respect to the multiplication. However if zero is excluded, the remaining system is an abelian group. More precisely the set $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ of non-zero complex numbers is closed under multiplication of complex numbers. The pair $(\mathbb{C}^*, \cdot)$ is an abelian group.

The set

$$S^1 = \{\mathbf{x} \in \mathbb{C} \mid |\mathbf{x}| = 1\}$$

is a subgroup of $(\mathbb{C}^*, \cdot)$, in particularly an abelian group with respect to the multiplication of complex numbers. A zero element of this group is $1 = (1, 0)$ and an inverse $(x, y)^{-1}$ of an element (x, y) is

$$(x, y)^{-1} = (x, -y).$$

The power z^n of a complex number is defined in the same way as a multiplicity in abelian groups, by induction,

$$z^n = \underbrace{z \cdot z \ldots \cdot z}_{n \text{ times}}.$$

Negative power is defined by

$$z^{-n} = (z^{-1})^n.$$

An element z of S^1 can be written in the form

$$z = (\cos 2\pi t, \sin 2\pi t),$$

where $t \in \mathbb{R}$. It can be shown that if $z = (\cos 2\pi t, \sin 2\pi t)$ and $w = (\cos 2\pi s, \sin 2\pi s)$, then

$$z \cdot w = (\cos 2\pi(t+s), \sin 2\pi(t+s).$$

In particular

$$z^n = (\cos 2\pi nt, \sin 2\pi nt).$$

Proposition 18.6. *Let $n \in \mathbb{Z}$ and let $f_n \colon S^1 \to S^1$ be defined by $p_n(z) = z^n$, where we treat $z \in S^1$ as a complex number. Then $\deg p_n = n$. In particular for every $n \in \mathbb{Z}$ there exists a mapping $f \colon S^1 \to S^1$ with $\deg f = n$.*

Proof. For $n = 0$ the mapping p_0 is a constant mapping, which certainly has degree 0. Also $p_{-1}(x, y) = (x, -y)$ is a reflection along the x-axis, which has degree -1, by Example 15.3. Since $p_{-n} = p_{-1} \circ p_n$, it is thus enough to consider only the case $n > 0$.

For every $k = 0, \ldots, n-1$ let

$$x_i = (\cos 2\pi ki/n, \sin 2\pi ki/n)$$

and define the path $\alpha_k \colon I \to S^1$ by

$$\alpha_k(t) = (\cos(1-t)2\pi ki/n + t2\pi(k+1)i/n, \sin(1-t)2\pi ki/n + t2\pi(k+1)i/n).$$

By the Example 9.10

$$x = \overline{\sum_{k=0}^{n-1} \alpha_k}$$

is a generator of $H_1(S^1)$ and $x = \overline{\gamma}$, where

$$\gamma(t) = (\cos 2\pi t, \sin 2\pi t).$$

We have that $(p_n)_\sharp(\alpha_k) = \gamma$, hence

$$(p_n)_*(x) = \sum_{k=0}^{n-1} [\gamma] = n[\gamma] = nx.$$

□

Let $n > 1$ and suppose $f\colon S^{n-1} \to S^{n-1}$ be a continuous mapping. We define **the suspension** $\Sigma f\colon S^n \to S^n$ of f as follows. Write

$$S^n = \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid |x|^2 + |t|^2 = 1\}.$$

Assert

$$\Sigma f(x,t) = \begin{cases} (|x| \cdot f(x/|x|), t), & \text{if } x \neq 0, \\ (x,t), & \text{if } x = 0. \end{cases}$$

The geometric idea behind this formula is that for every $c \in [-1,1]$ the " slice " $\{x \in S^n | x = c\}$ is homeomorphic to S^{n-1} in a natural way, except for extreme cases $c = \pm 1$, where this set reduces to a point (north and south poles of S^n). Using this homeomorphism we define Σf to "look like " f on every slice. North and south poles are fixed points. The verification of continuity of Σf is left as an exercise to the reader.

Proposition 18.7. $\deg \Sigma f = \deg f$.

Proof. Recall from Example 16.13 that $(S^n; S^n_+, S^n_-)$, where

$$S^n_+ = \{\mathbf{x} \in S^n \mid x_{n+1} \geq 0\},$$

$$S^n_- = \{\mathbf{x} \in S^n \mid x_{n+1} \leq 0\}$$

are upper and lower hemispheres of S^n, is a proper triad. For the suspension we have, for definition that $\Sigma f(S^n_+) \subset S^n +$ and $\Sigma f(S^n_-) \subset S^n_-$. Also $\Sigma f | S^{n-1} = f$. By the naturality of the Mayer-Vietoris sequence of the proper triad $(S^n; S^n_+, S^n_-)$ we obtain a commutative diagram

$$\begin{tikzcd} H_n(S^n) \arrow[r, "\Delta\cong"] \arrow[d, "(\Sigma f)_*"] & H_n(S^{n-1}) \arrow[d, "f_*"] \\ H_n(S^n) \arrow[r, "\Delta\cong"] & H_n(S^{n-1}), \end{tikzcd}$$

where vertical mappings are isomorphisms. The claim follows. □

Corollary 18.8. *For every $n \geq 1$ and every $m \in \mathbb{Z}$ there exists $f\colon S^n \to S^n$ s.t. $\deg f = m$. In particular the set $[S^n, S^n]$ is infinite (but countable).*

Proof. The first assertion follows from Propositions 18.6 and 18.7 by induction on n. Since homotopic mappings have the same degree, the mapping $[S^n, S^n] \to \mathbb{Z}$ defined by $\bar{f} = \deg f$ is well-defined (recall that $[S^n, S^n]$ means the set of all homotopy classes of cont. mappings $S^n \to S^n$). The first part of Corollary implies that this mapping is surjection. Since $\mathbb{Z}$ is infinite, also $[S^n, S^n]$ is infinite. It is countable by Corollary 5.11. □

It can be actually proved that the mapping $[S^n, S^n] \to \mathbb{Z}$ defined by $\bar{f} = \deg f$ is even bijective, i.e. mappings $f, g \colon S^n \to S^n$ are homotopic if and only if they have the same degree. The proof of this fact is too elaborate for this course.

As the last application we will prove the fundamental theorem of algebra. Recall that the mapping $p \colon \mathbb{C} \to \mathbb{C}$ (where $\mathbb{C}$ is the set of complex numbers) is called *a polynomial*, if there exists $n \in \mathbb{N}$ and fixed coefficients $a_0, \ldots, a_{n-1} \in \mathbb{C}$, $a_n \in \mathbb{C}, a_n \neq 0$, such that for all $z \in \mathbb{C}$ we have that

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0.$$

If $n \geq 1$, polynomial is not a constant mapping.

A root of the polynomial $p \colon \mathbb{C} \to \mathbb{C}$ is a complex number $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

Theorem 18.9. *Every non-constant polynomial $p \colon \mathbb{C} \to \mathbb{C}$ has at least one root.*

Proof. Suppose p is a non-constant polynomial that does not have roots. We may assume that p is of the form

$$p(z) = z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

for some $a_0, \ldots, a_{n-1} \in \mathbb{C}$. Since p does not have roots, the mapping $f_r \colon S^1 \to S^1$ defined by

$$f_r(z) = p(rz)/|p(rz)|$$

is well defined and continuous for all $r \geq 0$. Also the mapping $f \colon S^1 \times [0, \infty[\to S^1$,

$$f(z, r) = f_r(z) = p(rz)/|p(rz)|$$

is well-defined and continuous. For every $r > 0$ the restriction $f|S^1 \times [0, r]$ is a homotopy[10] between constant mapping f_0 and f_r. Hence $\deg f_r = \deg f_0 = 0$

[10] In the formal definition of the homotopy one requires the homotopy to be the mapping $F \colon X \times I \to Y$, where I is a unit interval $[0, 1]$, but it is clear that any interval suffices.

for all $r > 0$.

On the other hand let r be any real number such that

$$r > 1 + |a_0| + |a_a| + \ldots + |a_{n-1}|.$$

Then for z with $|z| = r$ we have

$$|z^n| = r^n = r \times r^{n-1} > (|a_0| + |a_a| + \ldots + |a_{n-1}|)|z^{n-1}|$$

$$\geq |a_{n-1}z^{n-1}| + \ldots + |a_1 z| + |a_0| \geq |a_{n-1}z^{n-1} + \ldots + a_1 z + a_0|.$$

It follows that for $0 \leq t \leq 1$ the polynomial $p_t(z) = z^n + t(a_{n-1}z^{n-1} + \ldots + a_1 z + a_0)$ has no roots in the set $\{z \mid |z| = r\}$. In particular the homotopy $H\colon S^1 \times [0,1] \to S^1$ defined by

$$H(z,t) = p_t(rz)/|p_t(rz)|$$

is well-defined. Hence $f_r = H(\cdot, 1)$ is homotopic to $p_n = H(\cdot, 0)$, $p_n(z) = z^n$. By the proposition 18.6 it follows that $\deg f_r = \deg p_n = n$.
Hence $n = 0$, so p must be constant polynomial. $\square$