Chapter 4

CW-complexes and cellular homology

4.1 The concept of a CW-complex

In the very beginning of this course we have introduced the notion of simplicial complex and the concept of the triangulation of spaces. Later we have generalized this ideas to the concept of Δ-complex. We have seen that this approach is very fruitful and practical for the study of topological properties of spaces. For instance Theorem 3.4.3 shows that the simplicial structure allows us to compute the homology of a polyhedron or a polyhedron pair.

We have also seen some drawbacks of this techniques. Simplicial complexes are too " inflexible" for practical purposes and many spaces admit too complicated, not intuitive and " messy " triangulations in terms of simplicial complexes, that are useless for concrete calculations. Δ-complexes are a little better in this sense, but still rely on the linear, simplicial concepts, which are not purely topological in nature. For instance simplices are identified by affine homeomorpisms and all geometrical simplices have " sides " and " vertices " although some of them might be glued together.

The concept of a CW-complex is yet another generalization, which is entirely topological in nature and is much more "flexible". To see where the main idea for the generalization comes from recall the following properties of Δ-complexes, or, to be precise, their polyhedrons

(Lemma 1.3.2):

There is a collection of characteristic mappings $f_\sigma \colon \Delta_n \to |K|$ such that
1) The restriction of f_σ on the interior $\operatorname{int}\Delta_n$ is a homeomorphism to its image.
2) The image of the boundary $\partial\Delta_n$ under f_σ is a finite union of the interiors of the geometric simplices of smaller dimensions (i.e. $f'_\sigma(\operatorname{int}\Delta_m)$ for proper faces $\sigma' < \sigma$, $m < n$).
3) Characteristic mappings determine the topology of $|K|$ - in fact it is co-induced by the collection of all characteristic mappings.
4) $|K|$ is a disjoint union of geometric interiors $f_\sigma(\operatorname{int}\Delta_n)$.

It is a well-known fact (proved in the very first section of this notes) that from a topological point of view the pair $(\Delta_n, \partial\Delta_n)$ is homeomorphic to the pair $(\overline{B}^n, S^{n-1})$. Once we substitute the former pair by the later pair and omit all references to simplicial structures, we arrive precisely at the notion of a CW-complex.

Definition 4.1.1. *Suppose X is a Hausdorff space. Suppose that for every $n \in \mathbb{N}$ a collection $\mathcal{A}_n = \{f_\alpha \colon \overline{B}^n \to X\}$ of continuous mappings is given.Let*

$$\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n.$$

The pair $(X, \mathcal{A})$ is called a CW-complex if the following conditions are satisfied.
1) For every $\alpha \in \mathcal{A}$ the restriction of f_α to the interior B_n is an injection.
2) X is a disjoint union of the sets $f_\alpha(B_n)$, $\alpha \in \mathcal{A}$.
3) The topology of X is co-induced by the collection of mappings $\mathcal{A}$ i.e. a subset $A \subset X$ is open(closed) in X if and only if $f_\alpha^{-1}(A)$ is open(closed) for all $\alpha \in \mathcal{A}$.
4) For each $n \in \mathbb{N}$ and $\alpha \in \mathcal{A}_n$ the set $f_\alpha(S^{n-1})$ is contained in the finite union of the sets of the form $f_\beta(B^m)$ where $\beta \in \mathcal{A}_m$ for some $m < n$.

The mappings f_α are called characteristic mappings of the CW-complex $(X, \mathcal{A})$. Subspaces $f_\alpha(\overline{B}^n)$ are called (n-dimensional) **closed cells** of this CW-complex and are denoted also $\overline{e}_\alpha$. Subspaces $f_\alpha(B^n)$ are called

(n-dimensional) **open cells** and denoted e_α. Usually one abuses notation and calls the space X a CW-complex if it admits a collection $\mathcal{A}$ that makes it into one. Such collection however is by no means unique and the set of cells also depends on the chosen structure - see examples below.

Let us also elaborate on condition 4). Suppose $\alpha \in \mathcal{A}_n$. Then there exists a finite set $\mathcal{A}' \subset \bigcup_{m<n} \mathcal{A}_m$ such that

$$f_\alpha(S^{n-1}) \subset \bigcup \{\overline{e}_\beta \mid \beta \in \mathcal{A}'\}.$$

We can continue by induction and apply the same property to every set $\overline{e}_\beta$ (or rather its boundary).Continuing in this way we see that property 4) implies that for every $\alpha \in \mathcal{A}_n$ here exists a finite set $\mathcal{A}' \subset \bigcup_{m<n} \mathcal{A}_m$ such that

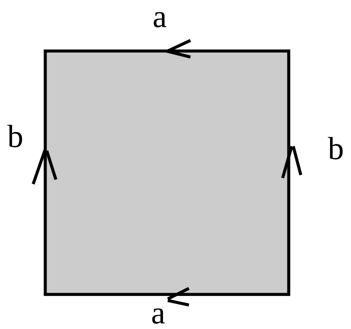
$$f_\alpha(S^{n-1}) \subset \bigcup \{e_\beta \mid \beta \in \mathcal{A}'\}.$$

Examples 4.1.2. *1) It follows from the discussion above that a polyhedron of a Δ-complex can be given a natural CW-structure - provided we know it is Hausdorff. This is always a case, but we will postpone the proof until later, when we will introduce alternative definition of the CW-complex that does not have Hausdorff assumption a priori.*

2) Consider a sphere S^n. We already know that it is a polyhedron of a Δ-complex - for instance obtained from two n-simplices glued together over a boundary. Since S^n is clearly Hausdorff, this defines immediately a CW-structure on S^n. However there is a simplier CW-structure, which does not come from any Δ-complex (unless $n = 1$). To define it recall that the quotient space obtained from $\overline{B}^n$ by collapsing the boundary S^{n-1} to a point is homeomorphic to S^n (see exercise 3.42). Hence there exists a quotient mapping $f\colon \overline{B^n} \to S^n$ that maps S^{n-1} onto a point, say $-e_{n+1} = (0, \dots, 0, -1)$ and B^n injectively onto $S^n \setminus \{-e_{n+1}\}$. Define also $g\colon \overline{B}^0 = \{0\} \to S^n$ to be a constant mapping $g(0) = e_{n+1}$. Now the collection $\{f, g\}$ defines a CW-structure on X. All the properties are easily seen to be satisfied by construction except perhaps property 3). Suppose $A \subset S^n$ is such that $f^{-1}(A)$ is closed in $\overline{B}^n$ and $g^{-1}(A)$ is closed in $\{0\}$. But f is a quotient mapping by construction (alternatively - because S^n is Hausdorff, $\overline{B}^n$ is compact and f is a continuous surjection) , so already condition $f^{-1}(A)$ is closed implies that A is closed.

Notice that this structure has only two closed cells and only two open cells. If $n > 1$ this structure does not come from any Δ-complex, since otherwise there has to be at least one $n-1$-dimensional cell.

3) Consider a torus T, which is obtained from a square by gluing horizontal and vertical sides preserving the natural orientation. Now T has a natural CW-structure - all corner points correspond to the only 0-cell, a and b (see the picture) are 1-cells and the image of the whole square in T after identification is the only 2-cell.



Notice that this is simpler as the Δ-complex structure, that had 2 2-simplices and also a diagonal as a 1-simplex. This is another example of efficiency of CW-approach. In a similar manner one can define a natural CW-structures on a Mobius band or Klein's bottle.

If $\mathcal{A}$ is finite, then the condition 3) is redundant. In fact the following is true.

Lemma 4.1.3. *Suppose X is a space and $\mathcal{A}$ collection of mappings each defined on some $\overline{B}^n$. Suppose conditions 1), 2) and 4) as above are satisfied. Then $(X, \mathcal{A})$ is a CW-complex if and only if one of the following conditions is satisfied*
3') The topology of X is coherent with the collection $\{\overline{e}_\alpha\}$ of all closed cells. In other words a subset $A \subset X$ is closed (or open) in X if and only if $A \cap \overline{e}_\alpha$ is closed (or open) in $\overline{e}_\alpha$.
3")The mapping

$$f = \sqcup f_\alpha \colon \sqcup_{\alpha \in \mathcal{A}} \overline{B}^{n_\alpha} \to X$$

is a quotient mapping. (Remark: $\sqcup$ denotes disjoint topological union).

In particular if $\mathcal{A}$ is finite and conditions 1), 2) and 4) are satisfied, then $(X, \mathcal{A})$ is a CW-complex.

Proof. Exercise 4.1. □

Suppose $(X, \mathcal{A})$ is a CW-complex, $Y \subset X$ and

Lemma 4.1.4. *Suppose $(X, \mathcal{A})$ is a CW-complex and $Y \subset X$. Then there exists $\mathcal{B} \subset \mathcal{A}$ such that $(Y, \mathcal{B})$ is a subcomplex of $(X, \mathcal{A})$ if and only if Y satisfies the following condition:*
Suppose $\alpha \in \mathcal{A}$ is such that $Y \cap e_\alpha \neq \emptyset$. Then $\overline{e}_\alpha \subset Y$.

Moreover in this case $\mathcal{B}$ is unique and Y is closed in X.

Proof. Suppose $(Y, \mathcal{B})$ is a subcomplex of $(X, \mathcal{A})$. Suppose $\alpha \in \mathcal{A}$ is such that there exists $y \in Y \cap e_\alpha$. Since $(Y, \mathcal{B})$ is itself a CW-complex there exists unique $\beta \in \mathcal{B}$ such that $y \in e_\beta$. Hence in particular $e_\alpha \cap e_\beta \neq \emptyset$. By disjointness $\alpha = \beta$. In particular $\alpha \in \mathcal{B}$, so

$$\overline{e}_\alpha \subset Y.$$

This proves that subcomplex satisfies the condition and also shows that $\mathcal{B}$ is unique - it is exactly the collection of those $\alpha \in \mathcal{A}$ with the property

$$e_\alpha \cap Y \neq \emptyset.$$

Conversely suppose $Y \subset X$ satisfies the condition above. Define

$$\mathcal{B} = \{\beta \in \mathcal{A} \mid e_\beta \cap Y \neq \emptyset\}.$$

Since $\overline{e}_\beta \subset Y$ for all $\beta \in \mathcal{B}$, every f_β restricts to a mapping $\overline{B}^n \to Y$ and Y is a disjoint union of open cells $e_\beta, \beta \in \mathcal{B}$ by constuction. Also every characteristic mapping is injection in interior of $\overline{B}^n$, since the same is true in the whole X.

Suppose $C \subset Y$ and $\alpha \in \mathcal{A}_n$. Then there exists a finite subset $\mathcal{A}' \subset \bigcup_{m<n} \mathcal{A}_m$ such that

$$C \cap f_\alpha(S^{n-1}) \subset \bigcup_{\beta \in \mathcal{A}'} C \cap e_\beta.$$

If $\beta \in \mathcal{A}'$ is not in $\mathcal{B}$, then $C \cap e_\beta = \emptyset$. Hence there exists a finite subset $\mathcal{B}' \subset \bigcup_{m<n} \mathcal{B}_m$ such that

$$C \cap f_\alpha(S^{n-1}) \subset \bigcup_{\beta \in \mathcal{B}'} C \cap e_\beta.$$

In particular if we assert $C = Y$ this shows that $(Y, \mathcal{B})$ satisfies condition 4).

Now suppose $C \subset Y$ is such that $C \cap \overline{e}_\beta$ is closed in $\overline{e}_\beta$ for all $\beta \in \mathcal{B}$. Suppose $\alpha \in \mathcal{A}_n$ is not in $\mathcal{B}$. Then

$$C \cap \overline{e}_\alpha = C \cap f_\alpha(S^{n-1}) = \bigcup_{\beta \in \mathcal{B}'} (C \cap \overline{e}_\beta \cap f_\alpha(S^{n-1}),$$

where $\mathcal{B}' \subset \mathcal{B}$ is finite. Now $\overline{e}_\beta \cap f_\alpha(S^{n-1})$ is a compact subset of Hausdorff space $\overline{e}_\beta$, so in particular closed in $\overline{e}_\beta$. Since $C \cap \overline{e}_\beta$ is closed in $\overline{e}_\beta$ by assumption, it follows that $C \cap \overline{e}_\beta \cap f_\alpha(S^{n-1}$ is closed in $\overline{e}_\beta$, hence also in X ($\overline{e}_\beta$ is closed in X since it is compact and X is Hausdorff). Since $\mathcal{B}'$ is finite it follows that $C \cap \overline{e}_\alpha$ is closed in X, as a finite union of closed sets. In particular it is closed in Y. Hence $(Y, \mathcal{B})$ satisfies condition 3). Also choosing $C = Y$ we see that Y is closed in X. $\square$

There is another way to define CW-complexes, which is often more convenient than the definition we gave. In order to present it let us first recall the notion of **the adjunction space**.

Suppose (X, A) is a topological pair, where A is closed in X. Suppose a continuous mapping $f\colon A \to Y$ is given. We define the adjunction space $X \cup_f Y$ as a quotient space of the disjoint union $X \sqcup Y$ under the equivalence relation generated by relations

$$a \sim f(a), a \in A.$$

In other words equivalence classes (i.e. the elements of $X \cup_f Y$) are singletons $\{x\}$, where $x \in X \setminus A$ and the sets

$$f^{-1}(y) \cup \{y\}, y \in Y.$$

Lemma 4.1.5. *Let $f\colon A \to Y$ be as above and let $p\colon X \sqcup Y \to X \cup_f Y$ be a canonical quotient projection. Then $p|X \setminus A$ is an open injection and $p|Y$ is a closed injection. In particular both restriction are embeddings, $p(X \setminus A)$ is open in $X \cup_f Y$ and $p(Y)$ is closed in $X \cup_f Y$*

Proof. Exercise 4.2. $\square$

In view of this result we will identify both Y and $X \setminus A$ with their p-images in $X \cup_f Y$ and consider both as subspaces of the adjunction space $X \cup_f Y$.

Now suppose Y is a topological space and $n \in \mathbb{N}$. Suppose $\mathcal{A}$ is a collection of some continuous mappings $f_\alpha \colon S^{n-1} \to Y$, $\alpha \in \mathcal{A}$. Denote

$$X = \sqcup_{\mathcal{A}} \overline{B}^n,$$

$$A = \sqcup_{\mathcal{A}} S^{n-1}.$$

Then A is closed in X and there is a continuous mapping

$$f = \sqcup_A f_\alpha \colon A \to Y.$$

Hence we can construct an adjunction space $Z = X \cup_f Y$. Every mapping $f_\alpha \colon S^{n-1} \to Y$ has a canonical extension $g_\alpha \colon \overline{B}^n \to Z$, defined as follows. Let $i_\alpha \colon \overline{B}^n \to X \sqcup Y$ be an embedding of the closed disk onto its α's copy in X and $p \colon X \sqcup Y \to Z$ be a canonical quotient projection. We define $g_\alpha \colon \overline{B}^n \to Z$ as a composition $p \circ i$. It follows straight from the construction that $g_\alpha | S^{n-1} = f_\alpha$, in other words g_α is an extension of f_α.
In this case we say that Z **is obtained from** Y **by attaching** n**-cells (via mappings** $\{g_\alpha\}_{\alpha \in \mathcal{A}}$**).**

More generally we say that a space Z is obtained from its closed subset Y by attaching n-cells via the set of mappings $\{h_\alpha \colon \overline{B}^n \to Z\}$ if there exists homeomorphism $h \colon X \cup_f Y \to Z$ which is identity on the subset Y. Here X and f are as above and $h_\alpha = h \circ g_\alpha$.

It is convenient to have some concrete characterization of this concept in terms of Z, Y and mappings h_α.

Lemma 4.1.6. *Suppose* (Z, Y) *is a topological pair and the collection* $\mathcal{A} = \{h_\alpha \colon \overline{B}^n \to Z\}$ *of continuous mappings is given. Then* Z *is obtained from* Y *by attaching n-cells via attaching mappings* $\{h_\alpha\}$ *if and only if the following conditions are satisfied.*
1) Z *is a disjoint union of* Y *and sets* $e_\alpha = h_\alpha(B^n)$, $\alpha \in \mathcal{A}$.
2) The topology of Z *is co-induced by mappings* h_α *and the inclusion* $Y \hookrightarrow Z$. *In other words* $A \subset Z$ *is open (close) if and only if* $A \cap Y$ *is open (closed) in* Y *and* $h_\alpha^{-1}(A)$ *is open (closed) in* $\overline{B}^n$ *for every* $\alpha \in \mathcal{A}$.
3) $Y \cap h_\alpha(\overline{B}_n) = h_\alpha(S^{n-1})$ *for all* $\alpha \in \mathcal{A}$.

Proof. Suppose Z is obtained from its closed subset Y by attaching n-cells via the set of mappings $\{h_\alpha \colon \overline{B}^n \to Z\}$. Properties 1) and 3) follow then straight from the definition of an adjunction space. The property 2)(together with 1) is clearly equivalent to $p \colon X \sqcup Y \to Z$ being a quotient mapping.

Hence the conditions are satisfied.

Conversely suppose the pair (X, Y) and mappings $\{h_\alpha \colon \overline{B}^n \to Z\, \alpha \in \mathcal{A}$ satisfy conditions. Let

$$X = \sqcup_{\mathcal{A}} \overline{B}^n$$

and define $h' \colon X \sqcup Y \to Z$ so that $h'|Y$ is identity onto $Y \subset Z$, and h' equals h_α on α's summond $\overline{B}^n$ of X. Conditions 1), 2) and 3) imply that h is then a surjective quotient mapping and the partition of $X \sqcup Y$ it defines equals precisely $X \cup fY$, where f is defined as above. $\square$

Suppose Z is obtained from Y by attaching n-cells. The sets $e_\alpha = f(B^n)$ are called open cells of the pair (Z, Y) and the sets $\overline{e}_\alpha = f(\overline{B}^n)$ are called closed cells of the pair (Z, Y). Notice that every open cell is open in Z and homeomorphic to B^n (via the restriction of h_α) and if Z is Hausdorff every closed cell is closed in X. It can be shown (Exercise 4.3) that in case Z is Hausdorff the set of open/closed cells depend only on the pair (Z, Y), not on the choice of attaching mappings.

Examples 4.1.7. *1. S^n is obtained from a point by attaching one n-cell:*
Let $f \colon S^{n-1} \to e_{n+1}$ be the unique constant mapping. Then $\overline{B}^n \cup_f \{e_{n+1}\}$ is nothing more but a quotient space of $\overline{B}^n$ with the boundary S^{n-1} identified to a point. We have seen above that this space is homeomorphic to a sphere S^n.
2. $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching an n-cell:
Let $f \colon S^{n-1} \to \mathbb{R}P^{n-1}$ be the canonical projection. Now the adjunction space $\overline{B}^n \cup_f \mathbb{R}P^n$ is homeomorphic to the quotient space of $\overline{B}^n$ with identifications $x \sim -x$ for the boundary points $x \in S^{n-1}$.
In the example **??** *above we have noticed that this is homeomorphic to $\mathbb{R}P^n$.*
3. Projective plane $\mathbb{R}P^2$ is obtained from the Mobius band by attaching a 2-cell:
Let X be a quotient space of the cylinder $S^1 \times I$ with identifications $(x, 1sim$ (-x, 1)$forallx$∈ S^1. In the exercise 1.26 we have shown that X is homeomorphic to the Mobius band. Let $f \colon S^1 \to X$ be the mapping $f(x) = [(x, 0)]$. We claim that $X \cup_f \overline{B}^2$ is homeomorphic to the projective plane $\mathbb{R}P^2$. Indeed X can also be represented as a quotient space of the annulus $\{x \in \overline{B}^2 \mid 1/2 \leq |x| \leq 1\}$ with identifications $x \sim -x$, $x \in S^1$. This is so because the annulus $\{x \in \overline{B}^2 \mid 1/2 \leq |x| \leq 1\}$ is homeomorphic to the cylinder $S^1 \to I$ via the homeomorphism

$$x \mapsto (x/|x|, 1/2|x| + 1/2)$$

and this homeomorphism takes $x \in S^1$ to $(x, 1)$.
*The attachment mapping becomes the mapping $x \mapsto (1/2)x$. It is clear that if ones attaches a 2-disk to the annulus $\{x \in \overline{B}^2 \mid 1/2 \leq |x| \leq 1\}$ along the boundary$\{x \in \overline{B}^2 \mid 1/2 = |x|$, one obtains a closed disk $\overline{B}^2$. Hence, taking identification on the other boundary in to account, it follows that $X \cup_f \overline{B}^2$ is homeomorphic to the quotient of the disk $\overline{B}^2$ with identifications $x \sim -x$ for $x \in nS^1$. This is known to be homeomorphic to $\mathbb{R}P^2$ (example **??**).*

Lemma 4.1.8. *Suppose Z is obtained from Y by attaching n-cells. If Y is Hausdorff, then also X is Hausdorff.*

Proof. Let $\mathcal{A} = \{h_\alpha \colon \overline{B}^n \to Z\}$ be the collection of attaching mappings. Define

$$U = Z \setminus \{h_\alpha(0) \mid \alpha \in \mathcal{A}\}.$$

Then U is open in Z by the condition 2) above, hence also the restriction $q = p|p^{-1} \colon p^{-1}U \to U$ is a quotient mapping (exercise 4.4). Define $r \colon U \to Y$ so that $r|Y = \mathrm{id}$ and the restriction of r on $\overline{e}_\alpha \setminus \{h\alpha(0)\}$ is defined so that

$$r_\alpha \circ h_\alpha(x) = h_\alpha(x/|x|) \text{ for all } x \in \overline{B}^n, x \neq 0.$$

It is easy to check that r is well-defined and $r \colon q$ is continuous. Since q is a quotient mapping, r is a continuous retraction $r \colon U \to Y$.

Now suppose $x, y \in Z, x \neq y$. If both x and y belong to Y they have disjoint neighbourhoods V_1 and V_2 in Y. Let $U_1 = r^{-1}V_1$ and $U_2 = r^{-1}V_2$. Then U_1 and U_2 are disjoint, $x \in U_1$ and $x \in U_2$ and U_1, U_2 are open in U, hence in Z. Thus x and y have disjoint neighbourhoods in Z.

Next suppose $x \in e_\alpha$ for some α. If $y \in e_\alpha$, then x and y have disjoint neighbourhoods in e_α, since e_α is homeomorphic to Hausdorff space B^n. Since e_α is open in Z, these are also disjoint neighbourhoods of x and y in Z.
If, on the other hand, $y \notin e_\alpha$, let $r' = |f^{-1}|(x)$ and choose $r \in]r', 1[$. Then $h_\alpha(B(0, r))$ is a neighbourhood of x and $Z \setminus \{h_\alpha \overline{B}(0, r)\}$ is a neighbourhood of y. Clearly these sets are disjoint. ☐

Notice that we have also proved the following. Suppose Z is obtained from Y by attaching n-cells. Then Y is so-called **neighbourhood retract** in Z i.e. there exists a neighbourhood U of Y in Z and a retraction $r \colon U \to A$.

Suppose $(X, \mathcal{A})$ is a CW-complex. For every $n \in \mathbb{N}$ define **the n-skeleton** X^n of X by

$$X^n = \bigcup_{m \leq n} \{\overline{e}_\alpha \mid \alpha \in \mathcal{A}_m\} = \bigcup_{m \leq n} \{e_\alpha \mid \alpha \in \mathcal{A}_m\}.$$

In other words X^n is precisely a union of all closed (or open) cells of dimension smaller or equal to n. The elements of X^0 (0-cells, which are just points) are called **vertices** of a CW-complex X.
Lemma 4.1.4 easily implies that X^n is a subcomplex of X, in particular is a CW-complex itself, with the set of characteristic mappings precisely

$$\mathcal{A}^n = \bigcup_{m \leq n} \mathcal{A}_m.$$

Also the same lemma implies that X^n is closed in X.
Clearly $X^{n-1} \subset X^n$ for all $n \in \mathbb{N}$. In case $n = 0$ we interpet X^{-1} as an empty set.

Proposition 4.1.9. *X^n is obtained from X^{n-1} by attaching n-cells, via mappings $\{f_\alpha\}_{\alpha \in \mathcal{A}_n}$.*

Proof. We have to check that the conditions of Lemma 4.1.6 are satisfied. Since X^n is a disjoint union of all open m-cells, $m \leq n$ and X^{n-1} is a disjoint union of open m-cells, $m < n$, it follows that X^n is a disjoint union of X^{n-1} and open n-cells of X. Also condition 4) of CW-complexes implies that $f_\alpha(S^{n-1}) = X^{n-1} \subset \overline{e}_\alpha$.
It remains to show the condition 2) of Lemma 4.1.6. Suppose $A \subset X^n$ is such that $A \cap X^{n-1}$ is closed in X^{n-1} and $f_\alpha^{-1} A$ is closed in $\overline{B}^n$ for all $\alpha \in \mathcal{A}_\setminus$. We want to prove A is closed in X^n. Since X^n is a CW-complex with the set of attaching mappings $\bigcup_{m \leq n} \mathcal{A}_m$, it is enough to prove that $f_\alpha^{-1} A$ is closed in $\overline{B}^m$ for $\alpha \in \mathcal{A}_m, m < n$. But in this case

$$f_\alpha^{-1} A = f_\alpha^{-1}(A \cap X^{n-1}),$$

which is close since by assumption $A \cap X^{n-1}$ is closed in X^{n-1} and f_α is continuous. $\square$

Previous result provides a reason for the next definition.

Definition 4.1.10. *Suppose X is a topological space and $(X^n)_{n \in \mathbb{N}}$ is a countable collection of its subsets such that $X = \bigcup_{n \in \mathbb{N}} X^n$ and*

$$X^0 \subset X_1 \subset \ldots \subset X^n \subset X^{n+1} \subset \ldots.$$

We say that $(X^n)_{n\in\mathbb{N}}$ is a **CW-filtration** *of X if*
1) The topology of X is coherent with the collection $(X^n)_{n\in\mathbb{N}}$, i.e. a subset $A \subset X$ is open/closed in X if and only if $X^n \cap A$ is closed in X^n for all $n \in \mathbb{N}$.
2) X^n is obtained from X^{n-1} by attaching n-cells for all $n \in \mathbb{N}$. Here $X^{-1} = \emptyset$, hence X^0 is a discrete space.

Proposition 4.1.11. *Suppose $(X, \mathcal{A})$ is a CW-complex. Then its skeletons $\{X^n\}_{n\in\mathbb{N}}$ form a CW-filtration of X.*

Proof. Condition 2) is proved in Proposition 4.1.9 above, so it remains to show that the topology of X is coherent with the collection $(X^n)_{n\in\mathbb{N}}$.
Suppose $A \subset X$ is such that $A \cap X^n$ is closed in X^n for every $n \in \mathbb{N}$. Let $\overline{e}_\alpha$ be a closed n-cell of X. Then

$$A \cap \overline{e}_\alpha = (A \cap X^n) \cap \overline{e}_\alpha$$

is closed in $\overline{e}_\alpha$. Since the topology of X is coherent with the collection of all closed cells, the claim follows. □

Next we want to prove the opposite claim - every space with a CW-filtration is a CW-complex in natural way. This provides another definition of a CW-complex, which is often more convenient in practise. Notice that in the definition of a CW-filtration we did not assume a priori that the space is Hausdorff.

Suppose X has a CW-filtration $(X^n)_{n\in\mathbb{N}}$. Denote by $\mathcal{A}_n$ the collection of attaching mappings of the pair (X^n, X^{n-1}). Define

$$\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n.$$

Proposition 4.1.12. *Suppose X has a CW-filtration $(X^n)_{n\in\mathbb{N}}$. Then $(X, \mathcal{A})$ (defined as in the previous paragraph) is a CW-complex.*

Proof. First of all we need to show that X is Hausdorff. By Lemma 4.1.8 and induction X^n is Hausdorff for every $n \in \mathbb{N}$. Also X^{n-1} is a neighbourhood retract of X^n i.e. there is an open subset U_n of X^n such that $X^{n-1} \subset U$ and there exists a retraction $r_n \colon U_n \to X^{n-1}$.
Now suppose $x, y \in X$, $x \neq y$. Choose $n \in \mathbb{N}$ such that $x, y \in X^n$. Let V_n

and W_n be disjoint neighbourhoods of x and y in X^n. We will construct by induction a chain

$$V_n \subset V_{n+1} \subset \ldots \subset V_m \subset V_{m+1} \subset \ldots,$$

$$W_n \subset W_{n+1} \subset \ldots \subset W_m \subset W_{m+1} \subset \ldots$$

such that V_m, W_m are open in X^m for all $m \geq n$ and disjoint. This is done as following. Suppose we already defined V_{m-1} and W_{m-1}. Then we assert

$$V_m = r_m^{-1} V_{m-1},$$

$$W_m = r_m^{-1} W_{m-1}.$$

Finally we put

$$V = \bigcup_{m \geq n} V_m,$$

$$W = \bigcup_{m \geq n} W_m.$$

Then V and W are open neighbourhoods of x and y in X and they are disjoint.

Clearly every attaching mapping is injective when restricted to the open ball B^n. Since X^n is obtained from X^{n-1} by attaching n-cells, it is disjoint union of all open n-cells and X^{n-1}. By induction it implies that X^n is a disjoint union of all open m-balls, $m \leq n$. Hence X is a disjoint union of open cells.

To check condition 3), let A be a subset of X such that $f_\alpha^{-1}(A)$ is closed in $\overline{B}^n$ for every $\alpha \in \mathcal{A}_n$, $n \in \mathbb{N}$. We need to prove that A is closed in X. It is enough to prove that $A \cap X^n$ is closed in X^n for every $n \in \mathbb{N}$ and we prove this by induction on n.
If $n = 0$ $A \cap X^0$ is closed in X^0 trivially, since X^0 is discrete, so its every subset is closed in it.
Suppose $B = A \cap X^{n-1}$ is closed in X^{n-1}. Then $(A \cap X^n) \cap X^{n-1} = B$ is closed in X^{n-1} and for every $\alpha \in \mathcal{A}_n$ we have

$$f_\alpha^{-1}(A \cap X^n) = f_\alpha^{-1}(A) \cap f_\alpha^{-1}(X^n) = f_\alpha^{-1}(A)$$

is closed in , $\overline{B}^n$ by assumption. Here we used the fact that the image of f_α lies entirely in X^n, so $f_\alpha^{-1}(X^n) = \overline{B}^n$.
Since X^n is obtained from X^{n-1} by attaching n-cells, this implies that $A \cap X^n$

is closed in X^n and we are done.

It remains to show the condition 4) of CW-complexes. Let C be a compact subset of X^n for some $n \in \mathbb{N}$. By the exercise 4.5a) C intersects only finitely many $n-1$-cell, so it contains in the union of finitely many open n-cell e_α and the set $C \cap X^{n-1}$. Since X is Hausdorff and C is compact, so it is closed, so $C \cap X^{n-1}$ is a compact subset of X^{n-1} and we may assume by induction that it can be covered by finitely many open cells of dimension at most $n-1$. We need only to check that the initial step in the induction is possible. But X^0 is discrete so its compact subsets are finite, hence can be covered by finitely many 0-cells. Thus we have shown that every compact subset of X^n can be covered by finitely many closed cell of dimension at most n. In particular for every $\alpha \in \mathcal{A}_n$ the set $f_\alpha(S^{n-1})$ is a compact subset of X^{n-1}, hence can be covered by finitely many m-cells, where $m < n$. □

Examples 4.1.13. *1. Now we can finally prove that $|K|$ is a CW-complex. Indeed n-skeletons $|K|^n$ form a CW-filtration of $|K|$. Details are left to the reader to fill in.*
In particular $|K|$ is always a Hausdorff space.

2. We have proved above that the projective n-plane $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching a single n-cell. Hence we have a CW-filtration

$$\mathbb{R}P^0 = \{0\} \subset \mathbb{R}P^1 = S^1 \subset \ldots \subset \mathbb{R}P^{n-1} \subset \mathbb{R}P^n.$$

Hence $\mathbb{R}P^n$ is an n-dimensional CW-complex, which has exactly one m-cell for every $m \leq n$.

4.2 Cellular homology

In the previous chapter we have shown that one can calculate the homology of a polyhedron using only simplicial structure, which in many cases is very useful computation technique. Now we would like to generalize this approach for CW-complexes.
By analogy on simplicial homology we should start with a complex $C(X)$ such that C_n is a free abelian group on the set of n-cells of a CW-complex X. However the next step - definition of a boundary operator is not so obvious now, since our cells do not have any " simplicial structure ", so we cannot talk about "faces" of a cell. It turns out that the best approach is to use singular homology theory which is already developed. For this we need

to prove some basic homological properties of CW-complexes.

Suppose $(X, \mathcal{A})$ is a CW-complex and $n \in \mathbb{N}$. Suppose $f_\alpha \in \mathcal{A}_n$ is an attaching mapping $f_\alpha \colon \overline{B}^n \to X^n$. By composing it with some (fixed) homeomorphism $\Delta_n \to \overline{B}^n$, we obtain a singular n-simplex $g_\alpha \in C_n(X^n)$. Moreover, its boundary belongs to $C_{n-1}(X^{n-1})$, hence there is a homology class $[g_\alpha] \in H_n(X^n, X^{n-1})$.

Lemma 4.2.1. *Suppose X is a CW-complex. Then*
a)

$$H_m(X^n, X^{n-1}) = \begin{cases} \textit{free abelian group on the set of all } n-\textit{cells of } X \textit{ if } m = n, \\ 0, m \neq n. \end{cases}$$

Moreover $\{[g_\alpha] \mid \alpha \in \mathcal{A}_n\}$ is a basis of $H_n(X^n, X^{n-1})$.
b) $H_k(X^n) = 0$ if $k > n$.
c) The inclusion $i \colon X^n \to X$ induces isomorphisms

$$i_* \colon H_k(X^n) \to H_k(X)$$

if $k < n$.

Proof. The proof of a) is very similar to the calculations made in the proof of Lemma 3.4.2, and it is thus left to the reader.

b) and c) follow from a) with the help of long exact sequence of the pair (X^n, X^{n-1}). Indeed, that sequence and a) imply that the homomorphism $i_* \colon H_k(X^{n-1}) \to H_k(X^n)$ induced by inclusion is an isomorphism when $k \neq n, n-1$.
In particular it follows that if $k > n$

$$H_k(X^n) \cong H_k(X^{n-1}) \cong H_k(X^{n-2}) \cong \ldots \cong H_k(X^0) = 0,$$

since X^0 is a discrete space and $k > 0$.
On the other hand if $k < n$ and $m \geq n$, then

$$i_* \colon H_k(X^n) \to H_k(X^m)$$

is an isomorphism. Since singular homology has compact carries, this implies c) - details are left as an the exercise 4.7. $\square$

Now we define **the cellular chain complex** $C^{CW}(X)$ of a CW-complex X as following. We assert

$$C_n^{CW}(X) = H_n(X^n, X^{n-1})$$

for all $n \in \mathbb{N}$. As a boundary operator $\partial\colon C_n^{CW}(X) \to C_{n-1}^{CW}(X)$ we use the boundary operator $\partial'\colon H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}) \to H_n(X^{n-2})$ from the long exact singular homology sequence of the triple (X^n, X^{n-1}, X^{n-2}). Recall that it can also be defined as a composition of the boundary operator $\partial\colon H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1})$ of the pair (X^n, X^{n-1}) and a mapping $j_*\colon H_{n-1}(X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$ induced by the inclusion $j\colon X^{n-1} \to (X^{n-1}, X^{n-2})$ (Exercise 2.25).

$$\begin{tikzcd}
 & & & & & 0 & \\
0 \arrow[drr] & & & H_n(X^{n+1}) \arrow[r, phantom, "\cong"] & H_n(X) \arrow[ur] & & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "j_*"] & & & & \\
H_{n+1}(X^{n+1}, X^n) \arrow[urr, "\partial"] \arrow[rrr, "\partial"] & & & H_n(X^n, X^{n-1}) \arrow[rrr, "\partial"] \arrow[drr, "\partial"] & & & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_*"] & \\
 & & & & 0 \arrow[ur] & &
\end{tikzcd}$$

First of all we need to show that $\partial' \circ \partial' = 0$. Consider $j_*\colon H_{n-1}(X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$ and $\partial\colon H_{n-1}(X^{n-1}, X^{n-2}) \to H_{n-2}(X^{n-2})$. These homomorphism are two consecutive mappings from the long exact sequence of the pair (X^{n-1}, X^{n-2}). By exactness their composition is zero. Hence also

$$\partial' \circ \partial' = j_* \circ \partial \circ j_* \circ \partial = 0.$$

It follows that (C^{CW}, ∂) defined as above is a chain complex. It will be called **the cellular chain complex** of the CW-complex X. The n-th homology of this complex will be denoted $H_n^{CW}(X)$.

By definition

$$H_n^{CW}(X) = Z_n^{CW}(X)/B_n^{CW}(X)$$

where $Z_n^{CW}(X) = \operatorname{Ker}\partial' = \operatorname{Ker}(j_* \circ \partial)$. From the long exact homology sequence of the pair (X^{n-1}, X^{n-2}) and Lemma 4.2.1a), it follows that j_* is an injection, thus

$$Z_n^{CW}(X) = \operatorname{Ker}(j_* \circ \partial) = \operatorname{Ker}\partial = \operatorname{Im}(j_*\colon H_n(X^n) \to H_n(X^n, X^{n-1})).$$

As above from the long exact homology sequence of the pair (X^n, X^{n-1}) and Lemma 4.2.1a), it follows that j_* is an injection, so defines an isomorphism

$$j_*\colon H_n(X^n) \cong Z_n^{CW}(X).$$

Now we are finally ready to prove that cellular homology is isomorphic to singular homology.

Proposition 4.2.2. *Suppose X is a CW-complex and $n \in \mathbb{N}$. Let $i\colon X^n \to X$ be an inclusion. Then the homomorphism $\rho_n = i_* \circ (j_*)^{-1}\colon Z_n^{CW}(X) \to H_n(X)$ is a surjection, whose kernel is exactly $B_n^{CW}(X)$.*
Hence ρ_n induces an isomorphism

$$H_n^{CW}(X) \cong H_n(X).$$

Proof. Since $j_*\colon H_n(X^n) \to Z_n^{CW}(X)$ is an isomorphism, to prove the first claim, it is enough to show that $i_*\colon H_n(X^n) \to H_n(X)$ is a surjection. Since $i_* = k_* \circ l_*$, where $k\colon X^{n+1} \to X$ and $l\colon X^n \to X^{n+1}$ are inclusions and k_* is an isomorphism by Lemma 4.2.1c), it is enough to show that l_* is a surjection. But this follows from the long exact sequence of the pair (X^{n+1}, X^n) and Lemma 4.2.1a).

Now $B_n^{CW}(X) = \operatorname{Im}\partial = \operatorname{Im}(j_* \circ \partial)$. Since j_* is an isomorphism onto $Z_n^{CW}(X)$, its inverse maps $B_n^{CW}(X)$ onto $\operatorname{Im}\partial \subset H_n(X^n)$, which, by exactness of the long exact homology sequence of the pair (X^{n+1}, X^n) equals $\operatorname{Ker} l_*$ as defined above. Since k_* is an isomorphism, $\operatorname{Ker} l_* = \operatorname{Ker} i_*$. It follows that $B_n^{CW}(X)$ is a kernel of ρ_n.

Now the last claim follows from the first and second claims by the first isomorphism theorem of the group theory. $\square$

Suppose X and Y are CW-complexes. A continuous mapping $f\colon X \to Y$ is called **cellular** if $f(X^n) \to Y^n$ for all $n \in \mathbb{N}$. Cellular mappings are analogues of simplicial mappings in the theory of simplicial complexes. Also the

analogue of the simplicial approximation theorem is true for CW-complex - it can be shown that any continuous mapping $f\colon X \to Y$ is homotopic to a cellular mapping, also we won't prove it in this course.
It is clear that a cellular mapping $f\colon X \to Y$ defines induced mappings $f_*\colon C_n^{CW}(X) \to C_n^{CW}(Y)$ for every $n \in \mathbb{N}$. Those constitute a chain mapping $f^{CW}\colon C^{CW}(X) \to C^{CW}(Y)$, hence define induced mappings $f_*^{CW}\colon H_n^{CW}(X) \to H_n^{CW}(Y)$.
It is easy to verify that the isomorphisms $\overline{\rho}_n\colon H_n^{CW}(X) \to H_n(X)$ and $\overline{\rho}_n\colon H_n^{CW}(Y) \to H_n(Y)$ commute with homomorphisms induced by cellular mappings, i.e. a diagram

$$\begin{tikzcd} H_n^{CW}(X) \arrow[r, "\overline{\rho}_n"] \arrow[d, "f_*^{CW}"] & H_n(X) \arrow[d, "f_*"] \\ H_n^{CW}(Y) \arrow[r, "\overline{\rho}_n"] & H_n(Y) \end{tikzcd}$$

is commutative. This can be also expressed by saying that homomorphisms $\overline{\rho}$ **are natural with respect to cellular mappings**. The precise verification of this fact is left to the reader.

Before going into the next example let us make the following observation. Suppose X is a CW-complex that has only one vertex (i.e. X^0 is a singleton). Then the boundary operator $\partial\colon C_1^{CW}(X) \to C_0^{CW}(X)$ is zero. This is because $C_0^{CW}(X) \cong \mathbb{Z}$, so if ∂ would not be zero, its image would be a non-trivial subrgoup of $\mathbb{Z}$, hence $H_0(X) \cong H^{CW}(X)$ would be a finite group $\mathbb{Z}_n$ for some $n \in \mathbb{N}$, in particular it is not free abelian group or it is a trivial singleton group (in case $n = 0$. On the other hand we know that $H_0(X)$ is always a free abelian group, which is not trivial, unless X is empty.
Hence $\partial\colon C_1^{CW}(X) \to C_0^{CW}(X)$ must be zero in this case.

Example 4.2.3. *Suppose $\mathcal{A}$ is a set and fix an integer $n \geq 1$. For every $\alpha \in \mathcal{A}$ take a copy S_α^n of a sphere S^n and let x_α be a "south pole" $e_{n+1} \in S_\alpha^n$. Let X be a space obtained from disjoint union of spheres S_α^n by identifying points x_α to one point x. Then X is a CW-complex - take as attaching mappings for every $\alpha \in \mathcal{A}$ a quotient mapping $f_\alpha\colon \overline{B}^n \to S_\alpha$ that maps S^{n-1} onto x and a constant mapping $g\colon \{0\} \to \{x\}$. Then X has $\mathcal{A}$ amount of open n-cells e_α, one 0-cell $\{x\}$ and no cells in other dimensions. The notation $X = \vee_{\alpha\in\mathcal{A}} S_\alpha^n$ is used and the resulted space is often referred to as the* **wedge sum** *of spheres S_α^n, $\alpha \in \mathcal{A}$.*
Hence $C^{CW}(X)$ is a chain complex with $C_n^{CW}(X) \cong \mathbb{Z}^{\mathcal{A}}$ a free abelian group

on the set of all n-cells of X, $C_0^{CW}(X) \cong \mathbb{Z}$ and other groups being trivial. If $n > 1$ this already implies that all boundary operators of $C^{CW}(X)$ are zero. If $n = 1$ this follows from the observation above, since X has only one vertex. Hence the cellular homology of X is

$$H_n^{CW}(X) \cong \mathbb{Z}^{\mathcal{A}},$$

with basis elements essentially attaching mappings $f_\alpha\colon \overline{B}^n \to S^n_\alpha$ (up to a homeomorphism $\Delta_n \to \overline{B}_n$),

$$H_0^{CW}(X) \cong \mathbb{Z}$$

and all other homology groups trivial. Hence the same is true also for singular homology.
Notice that in case α is a singleton and there is only one sphere, we obtain the CW-structure on S^n from the example 4.1.2. With thus structure the inclusion $i_\alpha\colon S^n \to X$ that maps S^n homeomorphically onto S^n_α is a cellular mapping and the map induced by it maps the generator of $H_n^{CW}(S^n) \cong \mathbb{Z}$ onto a generator $[f_\alpha]$. It follows that mappings $(i_\alpha)_*$ define an isomorphism

$$\oplus_{\alpha\in\mathcal{A}} i_\alpha\colon\ \oplus_{\alpha\in\mathcal{A}} H_n(S^n) \cong H_n(X).$$

Conversely let $p_\alpha\colon X \to S^n$ be a " projection to α-factor" of X i.e. a mapping that maps S^n_α onto S^n as identity and maps all the other S^n_β to a point S. Then p_α is a cellular mapping and induced mapping in n-th homology is an algebraic projection of direct sum $\mathbb{Z}^{\mathcal{A}}$ onto α-factor $\mathbb{Z}_\alpha$. This follows easily from the fact that $p_\alpha \circ i_\alpha = \mathrm{id}$, and $p_\alpha \circ i_\beta$ is a constant mapping for all $\beta \neq \alpha$. Indeed a constant mapping induces zero mapping in the n-th homology (because it factors through a singleton space, which has zero n-th homology), so we obtain equations

$$(p_\alpha)_* \circ (i_\alpha)_* = \mathrm{id},$$

$$(p_\alpha)_* \circ (i_\beta)_* = 0, \beta \neq \alpha.$$

It is known from algebra that (p_α) is then a projection.

In order to apply Theorem 4.2.2 one often needs to calculate the boundary operator, which might be difficult if you do it directly from the definition. Next we will show how to express boundary operator in terms of degrees of certain mappings $S^n \to S^n$.

Suppose $(X, \mathcal{A})$ is a CW-complex. The group $C_n^{CW}(X)$ has a basis $\{f_\alpha \mid \alpha \in \mathcal{A}_n\}$, so it is enough to calculate $\partial(f_\alpha)$ for every $\alpha \in \mathcal{A}_n$. Moreover, since

$C^{CW}_{n-1}(X)$ has a basis $\{f_\beta \mid \beta \in \mathcal{A}_{n-1}\}$, there exists unique finitely supported family $\{d_{\alpha\beta}\}_{\beta\in\mathcal{A}_{n-1}}$ of integer coefficients such that

$$\partial(f_\alpha) = \sum d_{\alpha\beta} f_\beta,$$

so it is enough to calculate coefficients $d_{\alpha\beta}$.

For every $\beta \in \mathcal{A}_{n-1}$ define a mapping $p_\beta \colon X^{n-1} \to S^{n-1}$ as following. On the complement of the open cell e_β we let p_β be a constant mapping that maps everything to a south pole $-e_{n+1}$. On the closed cell e_β we define p_β so that $p_\beta \circ f_\beta \colon \overline{B}^{n-1} \to S^{n-1}$ is a canonical quotient mapping g that induces a homeomorphism $\overline{B}^{n-1}/S^{n-2} \to S^{n-1}$ and maps S^{n-2} to a south pole $-e_{n+1}$. We can choose for instance g defined in explicitly in the Exercise 3.42.
In other words we identify the complement of e_β to a point. It is easy to see that p_β is continuous. Moreover the diagram

$$\begin{array}{ccc} H_{n-1}(\overline{B}^{n-1}, S^{n-2}) & & \\ \downarrow{\scriptstyle (f_\beta)_*} & \searrow{\scriptstyle g_*\cong} & \\ H_{n-1}(X^{n-1}, X^{n-2}) & \xrightarrow{(p_\beta)_*} & H_{n-1}(S^{n-1}, S) \end{array}$$

commutes, since it commutes on the space-level. Now S^{n-1} is an $n-1$-dimensional CW-complex, with S being its $n-2$-skeleton and g the only characteristic mapping in dimension $n-1$, so by Lemma 4.2.1a) $[g]$ is the generator of $H_{n-1}(S^{n-1}, S) \cong \mathbb{Z}$ (up to a homeomorphism $\Delta_{n-1} \to \overline{B}^{n-1}$). Moreover up to the same homeomorphism $[\mathrm{id}]$ is a generator of $H_{n-1}(\overline{B}^{n-1}, S^{n-2}) \cong \mathbb{Z}$ and $g_*[\mathrm{id}] = [g]$, so it follows that g_* above is an isomorphism.
It follows that $(p_\beta)_*$ takes a generator $f_\beta = (f_\beta)_*[\mathrm{id}]$ of $H_{n-1}(X^{n-1}, X^{n-2})$ to the generator of $H_{n-1}(S^{n-1}, S) \cong \widetilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$ and all other generators to zero. In other words $(p_\beta)_*$ can be thought of as a natural projection $C^{CW}_{n-1}(X) \to \mathbb{Z}$ to the β-summond.
The natural isomorphism $H_{n-1}(S^{n-1}, S) \cong \widetilde{H}_{n-1}(S^{n-1})$ above comes from a reduced exact homology sequence of the pair (S^{n-1}, S) and the fact that reduced homology groups of a singleton $\{S\}$ are trivial.
Suppose $x \in H_{n-1}(X^{n-1}, Xn-2)$. Since $H_{n-1}(X^{n-1}, X^{n-2})$ is free on the basis $\{f_\beta \mid \beta \in \mathcal{A}_{n-1}\}$, there exist unique integers n_β such that

$$x = \sum n_\beta f_\beta.$$

From the algebraic observations above it follows that up to the choice of the generator of $H_{n-1}(S^{n-1}, S)$ we have that $n_\beta = (_\beta)_*(f_\beta) \in H_{n-1}(S^{n-1}, S) \cong \mathbb{Z}$.

Now consider the following commutative diagram.

$$\begin{tikzcd}
H_n(\overline{B}^n, S^{n-1}) \arrow[r, "\partial\cong"] \arrow[d, "(f_\alpha)_*"] & \tilde{H}_{n-1}(S^{n-1}) \arrow[d, "(f_\alpha|)_*"] \arrow[dr, "\Delta_{\alpha\beta}"] & \\
H_n(X^n, X^{n-1}) \arrow[r, "\partial"] \arrow[dr, "\partial"] & \tilde{H}_{n-1}(X^{n-1}) \arrow[r, "p_\beta"] \arrow[d, "j_*"] & \tilde{H}_{n-1}(S^{n-1}) \\
& H_{n-1}(X^{n-1}, X^{n-2}) \arrow[ur, "p_\beta"] &
\end{tikzcd},$$

where $\Delta_{\alpha\beta} = p_\beta \circ f_\alpha| \colon S^{n-1} \to S^{n-1}$.

It follows that (up to isomorphisms)

$$d_{\alpha\beta} f_\beta = (p_\beta)_* \partial(f_\alpha) = (p_\beta)_* \partial(f_\alpha)_*[D] = \Delta_{\alpha\beta}(\partial[D]),$$

where D is a generator of $H_n(\overline{B}^n, S^{n-1})$ that corresponds to the identity mapping under a fixed chosen homeomorphism $\Delta_n \to \overline{B}_n$. It follows that $\partial(D)$ is a generator of $\tilde{H}_{n-1}(S^{n-1})$. In other words the coefficient $d_{\alpha\beta} \in \mathbb{Z}$ equals the degree of the mapping $\Delta_{\alpha\beta} \colon S^{n-1} \to S^{n-1}$ up to a sigh.

This makes sense also for $n = 1$, although we did not define a degree of a mapping $S^0 \to S^0$ but this can be done as for $n > 0$, since the reduced group $\tilde{H}_0(S^0)$ is isomorphic to $\mathbb{Z}$.
In practise however there is no need to apply this result in case $n = 1$, since the boundary operator $\partial_1 \colon H_1(X^1, X^0) \to H_0(X^0)$ is defined simply by the familiar formula

$$\partial_1(f_\alpha) = f_\alpha(1) - f_\alpha(0).$$

Also observe that although the calculation above identifies $d_{\alpha\beta}$ only up to a sigh, it makes no difference in practice, since we can always switch a basis element f_β of $C^{CW}_{n-1}(X)$ to $-f_\beta$.

Example 4.2.4. *Let us calculate the homology groups of the projective n-plane $\mathbb{R}P^n$ using cellular homology. Since $\mathbb{R}P^n$ is a CW-complex that has exactly one cell in every dimension $0 \le k \le n$, the complex $C^{CW}(\mathbb{R}P^n)$ contains groups $C^{CW}_k(\mathbb{R}P^n) = \mathbb{Z}[f_k] \cong \mathbb{Z}$ for $0 \le k \le n$ and trivial groups in all other dimensions. It remains to calculate the boundary operators. We have $\partial(f_k) = d_k f_{k-1}$, where d_k is a degree of the mapping $\Delta_k \colon S^{k-1} \to S^{k-1}$*

defined as above. Now what is this mapping?

Recall that $\mathbb{R}P^k$ can be identified with a quotient space of $\overline{B}^k$ under the equivalence relation $\sim$ with $x \sim -x$ for all $x \in S^{k-1}$. In fact then the characteristic mapping $f_k\colon \overline{B}^k \to \mathbb{R}P^k$ is precisely a quotient projection.
Now by the definition $\Delta = p_{k-1} \circ (f_k|)\colon S^{n-1} \to \mathbb{R}P^{n-1} \to S^{n-1}$ is defined as following. In the upper hemisphere B_+ of S^{n-1} it is a mapping that maps $x = (x_1, \dots, x_n) \in B_+$ first to the class of $[x_1, \dots, x_{n-1}] \in \mathbb{R}P^{n-1} = B^{n-1}/\sim'$ and then by p^{k-1} that identifies $\mathbb{R}P^{n-2}$ to a south pole. This corresponds to the composition of a standard homeomorphism $p\colon B_+ \cong B^{n-1}$ (projection) and a quotient mapping $g\colon \overline{B}^{n-1} \to S^{n-1}$ that identifies boundary S^{n-2} to a south pole. Moreover we can choose g as in the Exercise 3.42.
On the other hand in the lower hemisphere $B-$ of S^{n-1} we first apply mapping ι_toB_+, $\iota(x) = -x$, then mapping $g \circ p\colon B_+ \to S^{n-1}$ as above.
It follows that we can factor Δ as

$$\Delta = \beta \circ \alpha,$$

where $\alpha\colon S^{n-1} \to S^{n-1} \wedge S^{n-1}$ and $\beta\colon S^{n-1} \wedge S^{n-1} \to S^{n-1}$ are defined as following.
Let $i_1, i_2\colon S^{n-1} \to S^{n-1} \wedge S^{n-1}$ be inclusions onto the first and second summond.
The restriction of α on the upper hemisphere B_+ is $i_1 \circ g \circ p$ on the first copy of S^{n-1} in the wedge sum $S^{n-1} \wedge S^{n-1}$ as above, while the restriction of α on the lower hemisphere $B-$ is $i_2 \circ g \circ p \circ \iota$. Since both restrictions map equator $S^{n-1} = B_+ \cap B_-$ onto the base point $-e_{n+1}$ of $S^{n-1} \to S^{n-1} \wedge S^{n-1}$, it follows that α is well-defined and continuous.
Mapping $\beta\colon S^{n-1} \wedge S^{n-1} \to S^{n-1}$ is defined to be identity on both summonds. These definitions easily imply that indeed $\Delta = \beta \circ \alpha$, so $\Delta_* = \beta_* \circ \alpha_*$.

By example 4.2.3 there is a commutative diagram

$$\begin{tikzcd}
 & H_{n-1}(S^{n-1}) \oplus H_{n-1}(S^{n-1}) \arrow[dr, "id_* \oplus \mathrm{id}_*"] \arrow[dd, "\cong"] & \\
H_{n-1}(S^{n-1}) \arrow[ur, "(f_*, \iota \circ f_*)"] \arrow[dr, "\alpha"'] & & H_{n-1}(S^{n-1}) \\
 & H_{n-1}(S^{n-1} \wedge S^{n-1}) \arrow[ur, "\beta"'] &
\end{tikzcd},$$

where $f\colon S^{n-1} \to S^{n-1}$ is defined as an Exercise 3.42 with the aid of the mapping g and hence has degree 1.

Hence

$$\Delta_*(x) = (id_* \oplus id_*)(f_*, \iota_* \circ f_*)(x) = (id_* \oplus id_*)(x, (-1)^n x) = x + (-1)^n x =$$

$$= \begin{cases} 2x \text{ when } n \text{ is even} \\ 0, \text{ when } n \text{ is odd,} \end{cases}$$

where we used the fact that $\deg \iota = (-1)^n$ *(Exercise 3. 28). Hence* $\deg \Delta = 2$ *when n is even and 0 is n is odd.*

Thus the cellular chain complex of $\mathbb{R}P^n$ is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 0} \dots \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \longrightarrow 0$$

if n is even and

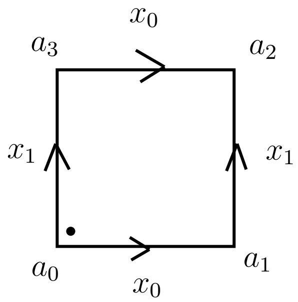
$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 2} \dots \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \longrightarrow 0$$

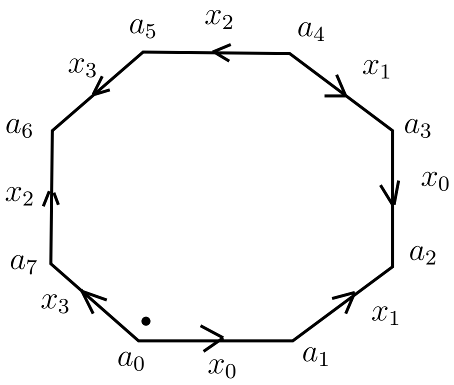
if n is odd.
From this we obtain immediately that

$$H_m(\mathbb{R}P_n) = \begin{cases} \mathbb{Z}, \text{ for } m = 0, \\ \mathbb{Z}_2, \text{ for } 0 < m < n \text{ if } m \text{ is odd }, \\ \mathbb{Z}, \text{ for } m = n \text{ if } n \text{ is odd }, \\ 0, \text{ otherwise.} \end{cases}$$

Compare this with the difficult and long calculation of the same groups in the Example 3.5.3, where we used Mayer-Viatoris sequence.

Example 4.2.5. The closed orientable surface M_g *of genus g is defined as follows. Take a(regular) polygon with $4g$ sides and denote its vertices by $a_0, \dots, a_{4g-1}$ in counter-clockwise fashion. Hence the faces of polygon are $[a_i, a_{i+1}]$ for $i = 0, \dots, 4g-1$. To obtain M_g we identify for every $i = 0, 4, \dots, 4k, \dots, 4(g-1)$, $k < g$ and every $i = 1, 5, \dots, 4k+1, \dots, 4g-3$, $k < g$ the side $[a_i, a_{i+1}]$ with side $[a_{i+3}, a_{i+2}]$ reversing direction, i.e. the point $ta_i + (1-t)a_{i+1}$, $t \in [0,1]$ with the point $ta_{i+3} + (1-t)a_{i+2}$. Here we denote $a_{4g} = a_0$.*
The cases $g = 1$ and $g = 2$ are illustrated in the picture below.





Notice that M_1 is a torus. We also define " degenerated case " $M_0 = S^2$.

It is not difficult to see that M_g is a compact 2-manifold for every $g \in \mathbb{N}$. On the other hand M_g is a 2-dimensional CW-complex with one 2-cell σ (corresponding to the whole space), $2g$ 1-cells, which we denote $x_0, x_1, \ldots, x_{2g-1}$ and one 0-cell, which corresponds to all vertices of the polygon (check that all vertices are identified with each other).

It follows that we can calculate the homology of M_g using cellular homology. The chain complex $C^{CW}(M_g)$ has non-zero groups only in dimensions $0, 1$ and 2, so the only boundary operators that need to be calculated are ∂_2 and ∂_1. Since M_g has only one vertex, ∂_1 must be zero. On the other hand

$$\partial_2(\sigma) = \sum_{0 \leq k < 2g} d_k x_k,$$

where d_k is a degree of the mapping $S^1 \to S^1$ which goes once around S^1 counter clockwise and then once in the opposite direction clockwise - because x_i occurs in the boundary of σ twice - in different directions. It is easy to check that this mapping has degree 0 (precise proof is left to the reader). Hence also $\partial_2 = 0$ and hence (cellular) homology of M_g is

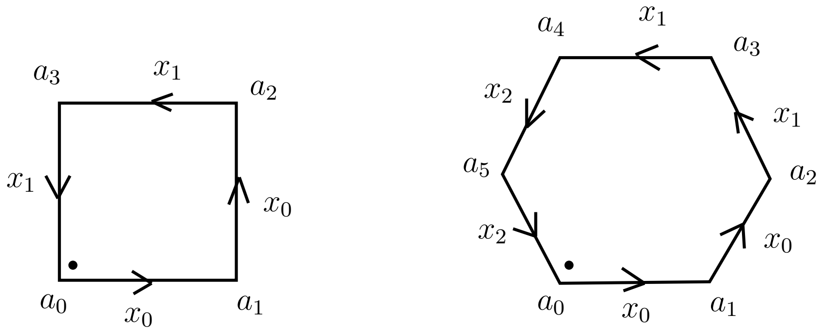
$$H_2(M_g) = \mathbb{Z},$$

$$H_1(M_g) = \mathbb{Z}^{2g},$$

$$H_0(M_g) = \mathbb{Z}.$$

Example 4.2.6. The closed non-orientable surface *N_g of genus g ($g \geq 1$) is defined as follows. If $g > 1$ take a(regular) polygon with $2g$ sides, with vertices $a_0, \ldots, a_{2g-1}$ (listed in counter-clockwise order). The faces of this*

polygon are $[a_i, a_{i+1}]$ *for* $i = 0, \dots, 4g-1$. *To obtain* N_g *we identify for every* $i = 0, 2, \dots, 2k, \dots, 42g-1)$, $k < g$ *the side* $[a_i, a_{i+1}]$ *with side* $[a_{i+1}, a_{i+2}]$ *the point* $ta_i + (1-t)a_{i+1}$, $t \in [0,1]$ *with the point* $ta_{i+3} + (1-t)a_{i+2}$. *Here we denote* $a_{4g} = a_0$.
The cases $g = 2$ *and* $g = 3$ *are illustrated in the picture below.*



We also define the " degenerated case " $N_1 = \mathbb{R}P^2$.

It is not difficult to see that N_g *is a compact* 2*-manifold for every* $g \geq 1$. *On the other hand* N_g *is a* 2*-dimensional CW-complex with one* 2*-cell* σ *(corresponding to the whole space),* g 1*-cells, which we denote* $x_0, x_1, \dots, x_{g-1}$ *and one* 0*-cell, which corresponds to all vertices of the polygon (check that all vertices are identified with each other).*

It follows that we can calculate the homology of N_g *using cellular homology. The chain complex* $C^{CW}(N_g)$ *has non-zero groups only in dimensions* 0, 1 *and* 2*, so the only boundary operators that need to be calculated are* ∂_2 *and* ∂_1. *Since* N_g *has only one vertex,* ∂_1 *must be zero. On the other hand*

$$\partial_2(\sigma) = \sum_{0 \leq k < g} d_k x_k,$$

where d_k *is a degree of the mapping* $S^1 \to S^1$ *which goes twice around* S^1 *because* x_i *occurs in the boundary of* σ *twice with the same orientation. The degree of this mapping is* 2 *(it is a mapping* $z \mapsto z^2$*, if complex numbers notation is used). Hence*

$$\partial_2(\sigma) = 2(x_0 + \ldots + x_{g-1}).$$

It follows that ∂_2 *is injection, whose image is the subgroup of* $C_1^{CW}(N_g)$ *generated by the element* $2(x_0 + \ldots + x_{g-1})$. *Since* $\{x_0, \dots, x_{g-1}\}$ *is a free basis of* $C_1^{CW}(N_g)$*, it follows that also* $\{x_0, \dots, x_{g-2}, x_0 + \ldots + x_{g-1}\}$ *is a free basis of* $C_1^{CW}(N_g)$. *Hence*

$$H_2(N_g) = \operatorname{Ker} \partial_2 = 0,$$

$$H_1(N_g) = (\mathbb{Z}[x_0]\oplus\mathbb{Z}[x_1]\oplus\ldots\oplus\mathbb{Z}[x_{g-2}]\oplus\mathbb{Z}[x_0+\ldots+x_{g-1}])/2\mathbb{Z}[x_0+\ldots+x_{g-1}] \cong \mathbb{Z}^{g-1}\oplus\mathbb{Z}_2.$$
$$H_0(N_g) = \mathbb{Z}.$$

Surfaces M_g and N_g play important role in the problem of classification of all compact 2-manifolds. In fact one can prove the following result.

Theorem 4.2.7. *Suppose X is a compact connected 2-manifold without boundary. Then there exists g such that X is homeomorphic to either M_g or N_g.*

For instance Klein's bottle is a compact connected 2-manifold. Since N_2 is the only orientable or non-orientable surface which has the same homology groups as the Klein's bottle, it follows from the Theorem 4.3.1 that Klein's bottle is homeomorphic to N_2. of course it it not difficult to see that also directly from the definitions of both spaces - just apply the good old "cut and paste" technique.

By the calculations above different spaces M_g and N_g all have different homology groups, so no two of them are of the same homotopy type. This observations combined with the result 4.3.1 leads to the following interesting fact.

Proposition 4.2.8. *Suppose X and Y are compact connected 2-manifold withut boundary. If X and Y are of the same homotopy type, they are actually homeomorphic.*

In other words the classification of compact 2-manifolds up to the homotopy types is the same as the classification up to the homeomorphism. In particular as a special case we obtain the following 2-dimensional version of the infamous **Poincare conjecture**:
A compact 2 manifold which has the same homotopy type as the sphere S^2 is actually homeomorphic to it.

The corresponding statement is actually true in all dimensions - a compact n-manifold which has the homotopy type of a sphere S^n is homeomorphic to S^n. This is known as **the generalized Poincare conjecture**. The cases $n = 1$ and $n = 2$ are classical results which were known for a long time. All cases $n \geq 4$ were proved up to 1982. The remaining case $n = 3$ turned out to be the most difficult one and it was exactly the case $n = 3$ which was known under the name **the Poincare conjecture** and remained unsolved for about a century. It was finally proved in 2003 by the eccentric Russian mathematician G. Perelman.

4.3 Classification of compact 2-mainfolds

The purpose of this Appendix is to outline the proof of the following result.

Theorem 4.3.1. *Suppose X is a compact connected 2-manifold without boundary. Then there exists g such that X is homeomorphic to either M_g or N_g.*

We will prove this theorem under assumption X is also a polyhedron of some simplicial complex i.e. has a triangulation. This is not really a restriction, since the following fact is true.

Proposition 4.3.2. *Every compact 2-manifold without boundary has a triangulation.*

However the proof of this Proposition is too difficult and will be omitted.

Definition 4.3.3. *Suppose K is a simplicial complex and $x \in |K|$.* **The simplicial neigbourhood** *of x $N(x)$ is defined to be the subcomplex of K generated by all the simplices that contain x.* **The link** *$Lk(x)$ of x is defined to be the subcomplex of $N(x)$ formed by the simplices of $N(x)$ that do not include x.*

Hence $\sigma \in K$ is in $N(x)$ if there exists $\sigma' \in K$ such that $x \in \sigma'$ and $\sigma < \sigma'$. The simplex $\sigma \in K$ is in $Lk(x)$ if $x \notin \sigma$ but there exists $\sigma' \in K$ such that $x \in \sigma'$ and $\sigma < \sigma'$. It is easy to see that

$$\mathrm{St}(x) = |N(x)| \setminus |Lk(x)|.$$

Lemma 4.3.4. *Suppose $x \in |K|$, K is a finite simplicial complex in a finite-dimensional vector space V. For every $y \in |N(x)|, y \neq x$ there exists precisely one $r(y) \in |Lk(x)|$ which lies on the half-line*

$$\{x + t(y - x) \mid t > 0\}.$$

The mapping $r\colon |N(x)| \setminus \{x\} \to |Lk(x)|$ (called the radical projection) is a continuous retract and homotopy inverse on the inclusion $i\colon |Lk(x)| \to |N(x)| \setminus \{\}$.

Proof. Since K is finite, $|K|$ is bounded, so there exists

$$t'(y) = \sup\{t \geq 0 \mid x + t(y - x) \in |N(x)|\}.$$

Then $t'(y) \geq 1 > 0$ and $r(y) = x + t(y - x) \in |N(x)|$, since $|N(x)|$ is closed. Let $\tau \in K$ be the carrier of $r(y)$. If $x \in \tau$, we can continue half-line $\{x + t(y - x) \mid t > 0\}$ further from $r(y)$, which contradicts the definition of $r(y)$. Hence $x \notin \tau$, so $r(y) \in |Lk(x)|$. To show the uniqueness of $r(y)$, it is enough to show that for all $t \in]0, t'[$ the point $z(t) = x + t(y - x)$ is not in $|Lk(x)|$. Let σ be the carrier of x. Since τ is a carrier of $r(y)$ and $r(y) \in |Lk(x)|$, there exists simplex σ' which contains x and has τ as one of its faces. Since σ is a carrier of x, σ is also a face of σ'. It follows that we can assume that σ' is a simplex spanned exactly by all the vertices of τ and σ. It is easy to see that the "open interval"

$$]x, r(y)[= \{x + t(r(y) - x) \mid t \in]0, 1[\}.$$

lies entirely in the interior of σ', hence does not intersect $|Lk(x)|$.

The verification of the continuity of r is left to the reader - it is enough to check it for sets $\sigma \setminus \{x\}$, where $\in K$ contains x. If $y \in |Lk(x)|$ clearly $r(y) = y$, so r is a retraction. Moreover linear homotopy $(y, t) \mapsto (1 - t)y + tr(y)$ is a well-defined continuous homotopy from identity to $i \circ r$. □

Proposition 4.3.5. *Suppose K, L are finite simplicial complexes and $f\colon |K| \to |L|$ is an embedding. Suppose $x \in |K|$ is such that $f(x) \in \operatorname{int} f(|K|)$. Then $|Lk(x)|$ and $|Lk(f(x))|$ have the same homotopy type.*

Proof. Let σ be a carrier of $f(x)$ in L and let U be open subset of L such that $f(x) \in U$. Then

$$f(x) \in U \cap \operatorname{St}(\sigma) \subset |N_L(f(x)|,$$

so $V = f^{-1}U \cap \operatorname{St}(\sigma)$ is an open neighbourhood of x that maps f into $|Nf(x)|$.

For every $\lambda \in]0, 1]$ define

$$\lambda|N(x)| = \{(1 - \lambda)x + \lambda y \| \; y \in |N(x)|\}.$$

Then $x \in \lambda|N(x)|$ and since $|N(x)|$ is finite, hence bounded for ever $\varepsilon > 0$ there exists $\lambda \in]0, 1]$ such that

$$|(1 - \lambda)x + \lambda y - x| = \lambda|x - y| < \varepsilon$$

for all $y \in |N(x)|$. Here $|\cdot|$ denotes some norm in $|K|$ induced by some embedding of $|K|$ in Euclidean space $\mathbb{R}^m$. In other words for every neighbourhood W of x there exists λ such that

$$\lambda|N(x)| \subset W.$$

In particular there exists λ such that

$$\lambda|N(x)| \subset V = f^{-1}U \cap \mathrm{St}(\sigma), \text{ in particular}$$

$$f(x) \in f(\lambda|N(x)|) \subset |N(f(x))|.$$

Notice that there is a homeomorphism $\lambda\colon |N(x)| \to \lambda|N(x)|$, $\lambda(y) = (1-\lambda)x + \lambda y$. In particular $\lambda(x) = x$.

Similarly $f(\lambda|N(x)|)$ contains an open neighbourhood of $f(x)$ in $|L|$ (since $\lambda|N(x)|$ contains an open neighbourhood of x in $|K|$ and f is a homeomorphism to its image), so there exists $\mu \in]0,1]$ such that

$$\mu|N(f(x)| \subset f(\lambda|N(x)|) \subset |N(f(x))|.$$

Iteration of the construction of λ also implies the existance of $\nu \in]0,1]$ such that

$$f(\nu|N(x)|) \subset \mu|N(f(x)| \subset f(\lambda|N(x)|) \subset |N(f(x))|.$$

We may clearly assume $\nu < \lambda$.

Next we define mappings $\phi\colon \mu|Lk(f(x)| \to f(\nu|Lk(x)|)$ and $\psi\colon f(\nu|Lk(x)|) \to \mu|Lk(f(x)|$ and show that they are homotopy inverses of each other. To define ϕ observe the following.

$$\mu|N(f(x)| \subset f(\lambda|N(x)|),$$

so in particular f^{-1} takes $\mu|Lk(f(x)|$ into $\lambda|N(x)| \setminus \{x\}$. We denote the restriction of f^{-1} defined as a mapping $f^{-1}\colon \mu|Lk(f(x)| \to \lambda|N(x)| \setminus \{x\}$ by the symbol f_1^{-1}. The radical projection $\lambda|N(x)| \setminus \{x\} \to \lambda|Lk(x)|$ defined as in the previous Lemma we denote by $r_\lambda\colon \lambda|N(x)| \setminus \{x\} \to \lambda|Lk(x)|$. Notice that previous Lemma also implies that r_λ is a homotopy equivalence. Next we scale $\lambda|Lk(x)| \to \nu|Lk(x)$ by $\nu \circ \lambda^{-1}$. Notice that this is essentially just a linear movement of the point $y \in \lambda|Lk(x)|$ on the half-line $\{(1-t)x + ty, t > 0\}$ that does not touch x. Now we are ready to define ϕ as a composition

$$\phi = f \circ (\nu \circ \lambda^{-1}) \circ r_\lambda \circ f_1^{-1}.$$

The mapping ψ has a slightly simplier description - it is simply composition $r_\mu \circ j$, where $j\colon f(\nu|Lk(x)|) \to \mu|N(f(x)| \setminus \{f(x)\}$ is an inclusion and $r_\mu\colon \mu|N(f(x)| \setminus \{f(x)\} \to \mu|Lk(f(x)|$ is a linear radical projection.

As a next step we construct a homotopy $F\colon \mu|Lk(f(x)| \times I \to \lambda|N(x)| \setminus \{x\}$ between the mapping $f_1^{-1}\colon \mu|Lk(f(x)| \to \lambda|N(x)| \setminus \{x\}$ and the mapping

$$k \circ (\nu \circ \lambda^{-1}) \circ r_\lambda \circ f_1^{-1},$$

where $k \hookrightarrow \nu|Lk(x)| \to \lambda|N(x)| \setminus \{x\}$ is an inclusion. In fact we only need to homotope $k \circ (\nu \circ \lambda^{-1}) \circ r_\lambda$ to the identity mapping, and it is easy to see that the linear homotopy will suffices for that.
Now f takes $\lambda|N(x)| \setminus \{x\}$ into $|N(f(x)| \setminus \{f(x)\}$, so $fF\colon \mu|Lk(f(x)| \times I \to |N(f(x)| \setminus \{f(x)\}$ is a well-defined homotopy between the inclusion $l\colon \mu|Lk(f(x)| \hookrightarrow |N(f(x)| \setminus \{f(x)\}$ and the mapping $\mu^{-1} \circ j \circ \phi$.
Since by the previous lemma $i \circ r_\mu\colon \mu|N(f(x)| \setminus \{f(x)\} \to \mu|Lk(f(x)| \hookrightarrow \mu|N(f(x)| \setminus \{f(x)\}$ is homotopic to identity, we see that $\mu^{-1} \circ j \circ \phi$ is homotopic to $\mu^{-1} \circ i \circ r_\mu \circ j \circ \phi = \mu^{-1} \circ i \circ (\psi\phi)$. On the other hand $\mu^{-1} \circ i$ is a homotopy inverse of $r_\mu \circ \mu$, so $\psi\phi$ is homotopic to $r_\mu \circ \mu \circ$, which is easily seen to be the identity of $\mu|Lk(f(x)|$. Hence $\psi\phi \approx \mathrm{id}$.

Now

$$\phi \circ \psi = f \circ (\nu \circ \lambda^{-1}) \circ r_\lambda \circ f_1^{-1} \circ r_\mu \circ j.$$

Since f^{-1} takes $\mu|N(f(x)|\setminus\{f(x)\}$ into $\lambda|N(x)|\setminus\{x\}$ we can subsitute $f_1^{-1} \circ r_\mu$ above by $f_2^{-1} \circ i \circ r_m u$, which is homotopic to f_2^{-1}, where $f_2^{-1}\colon \mu|N(f(x)| \setminus \{f(x)\} \to \lambda|N(x)|$ is a mapping defined by f^{-1}. Hence $\phi \circ \psi$ is homotopic to the composition $f \circ (\nu \circ \lambda^{-1}) \circ r_\lambda \circ f_2^{-1} \circ \circ j$. On the other hand $f_2^{-1} \circ \circ j = j' \circ f_\nu^{-1}$, where $f_\nu^{-1}\colon f(\nu|Lk(x)|) \to \nu|Lk(x)|$ is defined by f^{-1} and $j'\colon \nu|Lk(x)| \to \lambda|N(x)| \setminus \{x\}$ is an inclusion. Definitions easly imply that $(\nu \circ \lambda^{-1}) \circ r_\lambda \circ j'$ is identity of $\nu|Lk(x)|$. Hence in the end we obtain that $\phi\psi$ is homotopic to the identity of $f(\nu|Lk(x)|)$.

We have shown that $\mu|Lk(f(x))|$ and $f(\nu|Lk(x)|)$ have the same homotopy type. Since $\mu|Lk(f(x))|$ is homotopic to $|Lk(f(x))|$ and $f(\nu|Lk(x)|)$ is homotopic to $|Lk(x)|$, the claim is proved. □

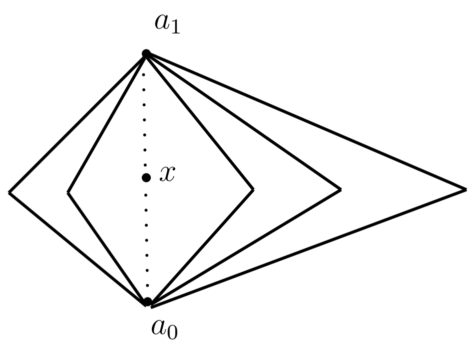
Corollary 4.3.6. *Suppose K is a finite simplicial complex such that $|K|$ is an n-manifold without boundary. Then $|Lk(x)|$ has the same homotopy type as S^{n-1} for every $x \in |K|$.*

Proof. Suppose $x \in |K|$ and let $f\colon U \to V$ be a homeomorphism, where U is an open subset of $\mathbb{R}^n$ and V is a neighbourhood of x in $|K|$. We may assume $0 \in U$ and $x = f(0)$. Let $r > 0$ be such that $\overline{B}^n(0,r) \subset U$. Now $\overline{B}^n(0,r)$ has a triangulation as $|K(\sigma)|$, where σ is an n-simplex. Now the restriction $f|\colon |K(\sigma)| \to K$ satisfies the conditions of the previous proposition for x, so $S^{n-1} \cong |Lk(0)| \approx |Lk(x)|$. □

Proposition 4.3.7. *Suppose K is a finite simplicial complex such that $|K|$ is a 2-manifold without boundary. Then*
1) $|K|$ is 2-dimensional and every point belongs to some 2-simplex.
2) Every 1-simplex of $|K|$ is a common face of exactly two 2-simplices in K.

Proof. Suppose $\dim K > 2$. Since K is finite, there exists then a maximal simplex $\sigma \in K$ of dimension $m > 2$. For any $x \in \operatorname{int}\sigma$ we have then $Lk(x) \cong S^{m-1}$. On the other hand by the previous corollary $LK(x)$ has the homotopy type of S^1. Hence S^{m-1} has the same homotopy type as S^1. But since $m > 2$, this contradicts Corollary 3.3.3.

1) now follows from 2), since every vertex belongs to some 1-simplex (otherwise it would be a discrete point, which would contradict the fact that $|K|$ is a 2-manifold).
Suppose τ is a 1-simplex, and let $\sigma_1, \ldots, \sigma_n$ are all different 2-simplices, that contain τ as a face. We have to prove that $n = 2$. Let x be a point in the interior of τ. It is easy to see that $Lk(x)$ is the union of all 1-faces of simplices $\sigma_1, \ldots, \sigma_n$, except for τ, which is homeomorphic to the union of n copies of the unit interval $[0, 1]$ with end points 0 and 1 identified together in all copies - see the picture below. Here a_0 and a_1 denote end points of τ.



It is easy to see that this implies that $H_1(Lk(x))$ is a free abelian group $\mathbb{Z}^{n-1}$ with $n-1$ free generators. On the other hand previous proposition implies that $H_1(X) = H_1(S^1) = \mathbb{Z}$. Hence $n-1 = 1$ i.e. $n = 2$. $\square$

Proposition 4.3.8. *Suppose K is a finite 2-dimensional simplicial complex such that $|K|$ is connected, $Lk(x)$ is connected for all $x \in K$ and every 1-simplex is a face of exactly two 2-simplices. Then there exists $k \geq 2$ such that $|K$— is homeomorphic to a space obtained from $2k$-polygon by identifying edges in pares.*

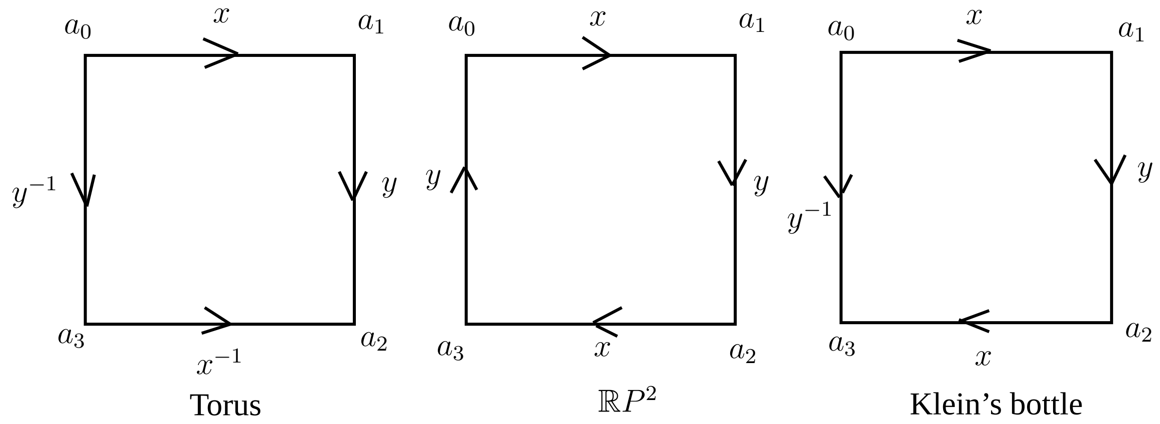
Proof. Start with any 2-simplex σ_1 of K and its fixed 1-face τ. Now there exists exactly one other simplex σ_2 which also has τ as a face. The union $\sigma_1 \cup \sigma 2$ is a subspace of $|K|$ homeomorphic to a square (because σ_1 and σ_2 intersect exactly at τ).

We continue this process by adding new triangles to already constructed subspace $\sigma_1 \cup \ldots \cup \sigma_m$ which looks like a polygon with possible some identification of exterior edges. Some edges might be identified with some other edge, which cannot be "interior" edge, since otherwise it would face 3 triangles. If some edge in $\sigma_1 \cup \ldots \cup \sigma_m$ is a face of only one simplex σ_i we can continue process and glue a new triangle σ_{m+1}.

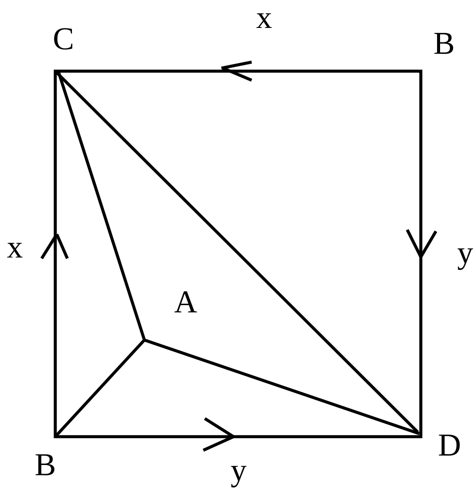
After a finite amount of steps this process stops and we obtain a subcomplex L such that $|L|$ is homeomorphic to a polygon with every edge identified with exactly one other edge. For this reason polygon must have even amount of faces.
The proposition is proved if we can show that in fact $L = K$. Make a counter-assumption, then there exist a vertex $a \notin L$. Since K is connected, there exists **edge-path** $a_0, \ldots, a_n$ connecting a to a vertex $b = a_n \in L$ (see exercise 9). Choosing the last vertex $a_i \notin L$ we may assume that a and $b = a_{i+1}$ span 1-simplex τ. We will obtain a contradiction by showing that $|Lk(b)|$ is not connected. Now b belongs to some 1-simplex of L, so there is a vertex c in $Lk(b) \cap L$. We will show that there is no edge-path in $Lk(b)$ from c to a. Suppose $c = c_0, \ldots, c_m = a$ is such edge-path. Again we may assume that $c = c_i$ is the last vertex from L in this path, so $d = c_{i+1}$ is not in L and c, d span a simplex $\tau' \in K$. Since $c, d \in |Lk(b)|$, there is a 2-simplex $\sigma \in K$ with vertices b, c, d. σ is not in L, since d is not. On the other hand construction of L shows that every 1-simplex of L is a face of exactly two simplices of L. In particular the simplex spanned by b, c is a face of two simplices in L and also a simplex σ. This contradicts the assumption. $\square$

Now it only remains to show that if a space is obtained from $2n$-polygon by identifying edges in pares, it is homeomorphic to M_g or N_g for some g.
Suppose $a_0, \ldots, a_{2n-1}$ are vertices of $2n$-sided polygon P and suppose X is obtained from P by identifying edges in pares. Now every edge (c, d) of P is identified in X with some other edge (c', d'). For each such pair of edges, denote both edges by the symbol like x_i for some index i. The reversed edge $(d, c) = (d', c')$ is then denoted by x_i^{-1}. Using these symbols we can describe X by the sequence of symbols x_i going through all the edges in order the vertices are listed - $(a_0, a_1), (a_1, a_2), \ldots, (a_i, a_{i+1}), \ldots, (a_{2n-1}, a_0)$.

For instance torus can be desribed by the sequence $xyx^{-1}y^{-1}$, the projective plane by $xyxy$, Klein's bottle by $xyxy^{-1}$ - see the pictures below.



More generally the space M_g, $g \geq 1$ corresponds to the sequence $x_1y_1x_1^{-1}y_1^{-1}\ldots x_2y_2x_2^{-1}y_2^{-1}$ and the space N_g, $g \geq 2$ - to the sequence $x_1x_1x_2x_2\ldots x_nx_n$. $M_0 = S^2$ corresponds to the sequence $xx^{-1}yy^{-1}$. To see this triangulate S^2 as a boundary of 3-simplex with vertices A, B, C, D and observe that simplices on this boundary can be fitted together in the following picture.



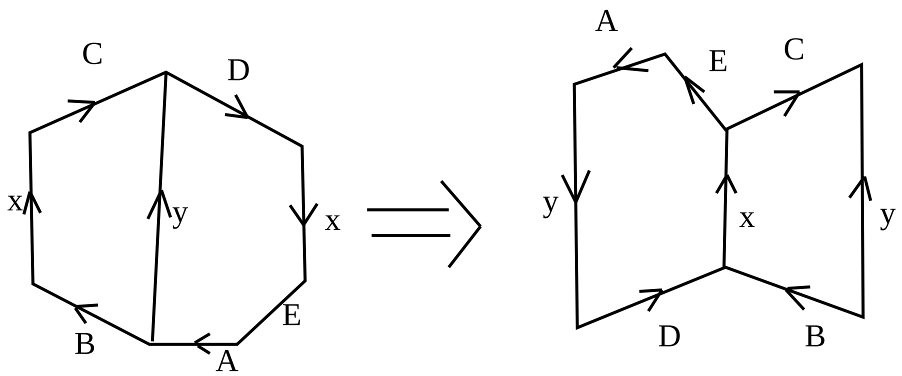
Proposition 4.3.8 shows that every polyhedron $|K|$ that satisfies assumptions of this proposition can be described by a finite sequence of symbols such as x or x^{-1}, in which each letter occurs exactly twice and there are at least two different letters. We will call such sequences **admissible**. Clearly each admissible sequence defines a quotient space of the $2k$-polygon. Some different admissible sequences define the same space up to a homeomorphism. In fact the next goal is to show that each admissible sequence can be modified

to a sequence corresponding to M_g or N_g, altering the space only up to a homeomorphism.
We first define **elementary rules** to alter admissible sequences.
By capital letters like A we will denote finite(possibly empty) subsequences of an admissible sequence. If $A = \alpha_1\alpha_2 \dots \alpha_n$, we denote by A^{-1} a sequence $\alpha_n^{-1} \dots \alpha_2^{-1}\alpha_1^{-1}$.

1)**Rule 1** : Replace $ABxCDxE$ by $AyDB^{-1}yC^{-1}E$, where y is a new symbol.
2) **Rule 2** : Replace $ABxCDx^{-1}E$ by $AyDCy^{-1}BE$.
3)**Rule 3** : Replace $Axx^{-1}B$ or $Ax^{-1}xB$ by AB, provided AB contains at least two different letters (each occurring twice, of course).

Proposition 4.3.9. *The application of Rules 1-3 to an admissible sequence gives a new admissible sequence whose corresponding space is homeomorphic to the space corresponding to the original sequence.*

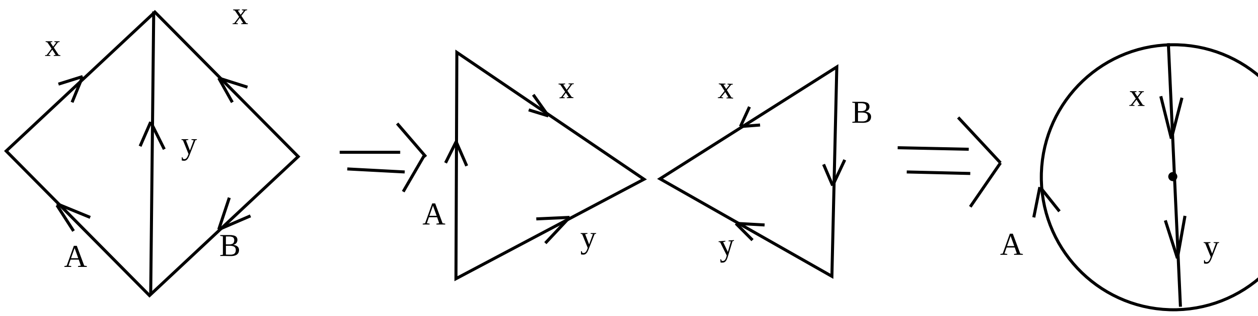
Proof. **Rule 1** : If BC and ADE are non-empty, the following picture shows the validity of the claim:



If BC is empty there is nothing to prove. If ADE is empty, replacement of $BxCx$ by $yB^{-1}yC^{-1}$ corresponds to the change of orientation on all edges and putting $x^{-1} = y$.

Rule 2 : This is proved in the same way, so the verification is left to the reader.

Rule 3 : By starting from the different vertex on the boundary, if possible, we may assume that both A and B represent at least 2 edges. The proof is then illustrated by the following picture:



□

Now we can start reducing admissible sequence to the " regular " representations. Each letter x in an admissible sequence occurs twice. These occurrences are called **a similar pair** if both have the same orientation i.e. if sequence is of the form $AxBxC$ or $Ax^{-1}Bx^{-1}C$ (in which case we can replace x^{-1} with new letter y. Occurrences are called **a reversed pair** if they have opposite orientations, i.e. sequence is of the form $AxBx^{-1}C$ or $Ax^{-1}BxC$.

Take an admissible sequence and apply the following 4 steps to it:
Step 1: Replace the sequence by the sequence AB, where A is of the form

$$x_1x_1x_2x_2\dots x_nx_n$$

and B contains only reversed pairs. This is done by induction and repeated application of the Rule 1, where we assume C is already of the form

$$x_1x_1x_2x_2\dots x_rx_r$$

:

$$CDxExF \to CyD^{-1}yE^{-1}F \to CzzDE^{-1}F.$$

Step 2: Two reversed pairs y, y^{-1} and z, z^{-1} are called interlocking if they occur in the sequence in the form $\cdots y \cdots z \cdots y^{-1} \cdots z^{-1} \cdots$.
Next we replace AB by the sequence ACD, where C is of the form

$$C = y_1z_1y_1^{-1}z_1^{-1}\dots y_kz_ky_k^{-1}z_k^{-1}$$

and D contains only non-interlocking reversed pairs. This is accomplished by several applications of the Rule 2, where we assume that E is already in the form like AC:

$$EFaGbHa^{-1}Ib^{-1}J \to EcGbHc^{-1}FIb^{-1}J$$
$$\to EcGdFIHc^{-1}d^{-1}J \to EeFIHGde^{-1}d^{-1}J$$
$$\to Eefe^{-1}f^{-1}FIHGJ.$$

Step 3: If A is non-empty, we can convert all interlocking reversed pairs into similar pairs. This uses Rule 1, but in reverse

$$Fxxaba^{-1}b^{-1}G \leftarrow Fyb^{-1}a^{-1}ya^{-1}b^{-1}G$$

$$\leftarrow Fyay^{-1}accG \leftarrow FyyddccG.$$

Step 4: Finally consider D which consists only of non-interlocking reversed pairs. Let the closest pair in D be the pair of edges x and x^{-1}. Suppose there is at least one symbol y between x and x^{-1}.If y^{-1} is not between x and x^{-1}, x and y form an interlocking pair, which is impossible. If, on the other hand y^{-1} is also between x and x^{-1}, then y and y^{-1} are closer then x and x^{-1}, which is also contradiction. Hence we can cancel out xx^{-1} by Rule 3, provided what remains contains at least two letters.

Let us look at what we have after all these steps. If A is non-empty and contains at least two similar pairs, we can get rid of all reversed pairs in Steps 3 and 4 hence in the end we arrive at the sequence of the form

$$x_1x_2\dots x_ny_g, g \geq 2,$$

which corresponds to the surface N_g, $g \geq 2$.
If A contains exactly one similar pair, we can perfome Step 3, so in case there were at least one interlocking pair, we arrive at the same type of sequence as above. If, on the other hand, there are no interlocking reversed pairs, at the Stage 4 we can get rid of all reversed pairs except for the last one, so we arrive at the sequence

$$xxyy^{-1}.$$

A single application of the Rule 1 in reverse changes this to $xy^{-1}xy^{-1}$, which corresponds to the projective plane N_1.
We are left with the case A is empty, i.e. there are no similar pairs. If there is at least one interlocking reversed pair, Step 4 allows us to get rid of all non-interlocking reversed pairs, hence we arrive at the sequence of the form

$$x_1y_1x_1^{-1}y_1^{-1}\dots x_gy_gx_g^{-1}y_g^{-1},$$

which corresponds to the surface M_g, $g \geq 1$.
Finally if there are only non-interlocking reversed pairs, we can cancel out of them, except for two, hence in the end we obtain sequence $xx^{-1}yy^{-1}$ or the sequence $xyy^{-1}x^{-1}$. But the latter defines the same space as $xx^{-1}yy^{-1}$, which we already noticed to be homeomorphic to $S^2 = M_0$ above.

This concludes the proof of the following proposition.

Proposition 4.3.10. *Suppose X is a topological space. Then the following conditions are equivalent:*
1) X is a compact connected 2-manifold.
2) $X = |K|$, where K is a 2-dimensional finite simplicial complex with following properties:
- Every 1-simplex of K is a face of exactly two 2-simplices.
- $|Lk(x)|$ has the homotopy type of S^1 for all $x \in K$.
3) $X = |K|$, where K is a 2-dimensional finite simplicial complex with following properties:
- Every 1-simplex of K is a face of exactly two 2-simplices.
- $|Lk(x)|$ is connected for all $x \in K$.
4) X is obtained from a $2k$-sided polygon ($k \geq 2$) by identifying edges in pairs.
5) X is homeomorphic to M_g or N_g for some g.

4.4 Exercises

1. Prove lemma 4.1.3

2. Prove lemma 4.1.5.

3. Suppose Z is obtained from Y by attaching n-cells. Show that the set of open cells depend only on the pair (Z, Y) (Hint: consider components of $X \setminus Y$). Assuming Z is Hausdorff show that the same is true for closed cells.

4. Suppose $p\colon X \to Y$ is a quotient mapping and $A \subset Y$ is open or closed. Show that $p|p^{-1}A\colon p^{-1}A \to A$ is a quotient mapping.

5. a) Suppose Z is obtained Y by attaching n-cells and C is a compact subset of Z. Then Z intersects only finitely many open cells of Z.
   b) Suppose X is a CW-complex and C is a compact subset of Z. Then there exists $n \in \mathbb{N}$ such that $C \subset X^n$.

6. Prove Lemma 4.2.1a).

7. Prove Lemma 4.2.1c), assuming the result is already proved for a finite-dimensional CW-complexes (Hint: use exercise 4.5 and the fact that singular homology has compact carries).

8. Suppose $(X, \mathcal{A})$ is a CW-complex and $(X_i, \mathcal{A}_i)$, $i \in I$ is a collection of subcomplexes of X. Prove that $(\bigcup_{inI} X_i, \bigcup_{inI} \mathcal{A})$ and $(\bigcup_{inI} X_i, \bigcup_{inI} \mathcal{A})$ are both subcomplexes of X.

9. Suppose X is a CW-complex and A is a path-component of X. Prove that A is a subcomplex of X.

10. Suppose X is a CW-complex. Prove that the following claims are equivalent:
    1) X is connected.
    2) X is path-connected.
    3) X^1 is path-connected.
    4) Every two vertices in X^0 can be joined by a path that lies in X^1.

11. Suppose K is a simplicial complex and a, b are vertices of K. **An edge-path** from a to b is a finite sequence of vertices $a = a_0, \ldots, a_n = b$ of K such that for all $i = 0, \ldots, n$ a_i and a_{i+1} belong to the same 1-simplex τ_i. In this case also the sequence $\tau_0, \ldots, \tau_{n-1}$ is also called an edge-path from a to b.
    Prove that $|K|$ is connected if and only if for every pair of vertices $a, b \in K$ there is an edge-path from a to b.

12. Suppose $g \in \mathbb{N}$ $(g \geq 1)$. Show that M_g (N_g) is a connected compact 2-manifold without boundary, which can be triangulated.

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