Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013
Exercise session 13 Solutions

1. Suppose M is an m-manifold and N is an n-manifold.
a) Suppose $m = n$ and M has no boundary. Prove that any continuous injection $f\colon M \to N$ is an open embedding, i.e. is a homeomorphism to its image $f(M)$, which is open in N.
b) Suppose $m > n$. Prove that there are no continuous injections $f\colon M \to N$.

Solution: a) It is enough to prove that every $x \in M$ has a neighbourhood U in M such that $f(U)$ is open in N and restriction $f|\colon U \to f(U)$ is a homeomorphism. Since N is an n-manifold there exists open neighbourhood V of $f(x)$ which is homeomorphic to a subset of $\mathbb{R}^n$ (open in $\mathbb{R}^n$ or $\mathbb{H}_n$, does not matter at this point). Since M is an n-manifold and f is continuous, there exists an open neighbourhood W of x in M which is homeomorphic to an open subset of $\mathbb{R}^n$ (here is where we need assumption that M has no boundary). Since $\mathbb{R}^n$ is locally compact, there exist neighbourhood U of x in W such that $\overline{U} \subset W$ and $\overline{U}$ is compact. This can be seen directly - choose $r > 0$ such that $B(x, r) \subset W$, then $U = B(x, r/2)$ is such that $\overline{U} \subset W$ and $\overline{U}$ is compact.
It is enough to prove that the restriction $f|U\colon W \to V$ is an open embedding.

The restriction $f|\colon \overline{U} \to V$ is a continuous injection from compact space to Hausdorff space(V is Hausdorff, since it is homeomorphic to subset of $\mathbb{R}^n$). By a well-known general topological fact $f|$ is homeomorphism to the image $f(\overline{U})$. Hence also its restriction $f\colon U \to f(U)$ to smaller subset U is a homeomorphism to the image $f(U)$ which is now a subset of $\mathbb{R}^n$. By invariance of domain theorem (version for $\mathbb{R}^n$) 17.6 the image $f(U)$ is open. We have shown that every $x \in M$ has a neighbourhood U in M such that $f(U)$ is open in N and restriction $f|\colon U \to f(U)$ is a homeomorphism.

b) Suppose $f\colon M \to N$ is a continuous injection.

Interior $\operatorname{int} M$ of any manifold is non-empty (we assume manifolds to be non empty). Indeed suppose $x \in M$. If $x \in \operatorname{int} M$, we are done.

Otherwise x has a chart $f\colon U \to f(U)$ where U can be assumed to be of the form

$$B(y,r) \cap \mathbb{H}_m = \{(y_1,\dots,y_m) \in B(y,r) \mid y \geq 0\}.$$

Then M contains an open set that is homeomorphic to

$$\{(y_1,\dots,y_m) \in B(y,r) \mid y > 0\},$$

which is a non-empty open subset of M.

Hence we may fix $x \in \operatorname{int} M$. Let V be a neughbourhood of $f(x)$ which is homeomorphic to the subset of $\mathbb{R}^n$. Since $x \in \operatorname{int} M$ we can find a small neighbourhood U of x which is homeomorphic to an open subset of $\mathbb{R}^n$ $f(U) \subset V$. Since V is a subset of $\mathbb{R}^n$ and $n < m$, V can be considered a subset of $\mathbb{R}^m$, since we have agreed to consider $\mathbb{R}^n$ a subset of $\mathbb{R}^m$ (vectors with $m-n$ last coordinates equal to zero). Thus we consider a restriction $f|\colon U \to V$ as a mapping $f\colon U \to \mathbb{R}^m$. Now U is an open neighbourhood of $\mathbb{R}^m$ and f is a continuous injection. Since U is an open neighbourhood of $\mathbb{R}^m$, it is an m-manifold without boundary (strictly speaking we need a) of the next exercise to know that), so by a) $f\colon U \to \mathbb{R}^m$ is an open embedding to the image. But this is contradiction, since $f(U)$ is a subset of $\mathbb{R}^n$, which has no interior in $\mathbb{R}^m$, so cannot be open in $\mathbb{R}^m$. Thus f cannot exist.

Remark: In fact we have proved a) under assumption "every point of M belongs to the interior i.e. has a neighbourhood homeomorphic to the open subset of $\mathbb{R}^n$", which is certainly true for any open subset U of $\mathbb{R}^n$. Hence we do not need results of the next exercise (which tells us, among other things, that assumption "every point of M belongs to the interior" is equivalent to "has no boundary".

2. Suppose M is an n-manifold.
   a) Prove that boundary ∂M and interior $\operatorname{int} M$ are disjoint.
   b) Prove that the interior $\operatorname{int} M$ is open in M and itself is an n-manifold without boundary.
   c) Prove that the boundary ∂M is closed in M and is an $(n-1)$-manifold without boundary.

Solution: a) Let us make a counter-assumption - there exists $x \in M$ which both interior and boundary point. Then there exists a chart

$f\colon U \to f(U)$, where U is open neighbourhood of $\mathbb{R}^n$ and $x \in f(U)$ as well as a chart $g\colon V \to f(V)$, where U is open neighbourhood of $\mathbb{H}_n$ and $x = g(y)$ for some

$$y \in V \cap \{z \in \mathbb{R}^n \mid z_n = 0\}.$$

Both $f(U)$ and $g(V)$ are neighbourhoods of x in M, so their intersection $W = f(U) \cap g(V)$ is also a neighbourhood of x. Restrictions $f|\colon f^{-1}(W) \to W$ and $g|\colon g^{-1}(W) \to W$ are homeomorphisms, so $g|^{-1} \circ f|\colon f^{-1}W \to g^{-1}W$ is also a homeomorphism, between open subset $f^{-1}W$ of $\mathbb{R}^n$ and a subset $g^{-1}W$ of $\mathbb{R}^n$. By invariance of domain 17.6. (version for $\mathbb{R}^n$) subset $g^{-1}W$ must also be open in $\mathbb{R}^n$. But it is not - it is a subset of

$$\mathbb{H}_n = \{x \in \mathbb{R}^n | x_n \geq 0\}$$

and contains at least one point y (defined above) that lies on the topological boundary

$$\{z \in \mathbb{R}^n \mid z_n = 0\}$$

of $g^{-1}W$ (w.r.t to $\mathbb{R}^n$). Thus we obtain a contradiction.

b) Suppose x belongs to $\operatorname{int} M$. Then, by definition x, has a neighbourhood U homeomorphic to an open subset of $\mathbb{R}^n$. But then all points of U also have a neighbourhood, namely U, which is homeomorphic to an open subset of $\mathbb{R}^n$. Hence $U \subset \operatorname{int} M$. Since this is true for every $x \in \operatorname{int} M$, $\operatorname{int} M$ is open in M. Also, since U above is subset of $\operatorname{int} M$, we see that every point of $\operatorname{int} M$ has a neighbourhood U homeomorphic to an open subset of $\mathbb{R}^n$. Thus $\operatorname{int} M$ is itself n-manifold, such that every its point belongs to its interior. By a) this is equivalent to $\operatorname{int} M$ not having boundary, as a manifold.

c) By a) boundary of M is the complement of $\operatorname{int} M$ and by b) $\operatorname{int} M$ is open in M, so ∂M is closed, being a compliment of an open set.

Suppose $x \in \partial M$. Let $f\colon U \to f(U)$ be a chart around x, such that $x = f(y)$, where

$$y \in U \cap \{z \in \mathbb{R}^n \mid z_n = 0\}.$$

Let $V = U \cap \{z \in \mathbb{R}^n \mid z_n = 0\} = U \cap \mathbb{R}^{n-1}$. Then V is open in $\mathbb{R}^{n-1}$. For every $w \in f(V)$ the chart $f\colon U \to f(U)$ is a chart around w such that $w = f(v)$ for some

$$v \in U \cap \{z \in \mathbb{R}^n \mid z_n = 0\}.$$

Hence $f(V) \subset f(U) \cap \partial M$.

Conversely, a point $w \in f(U) \setminus f(V) = f(U \setminus V)$ belongs to the interior of M, since $U \setminus V$ is an open subset of $\mathbb{R}^n$, so $f| \colon U \setminus V \to f(U \setminus V)$ is a chart of any point w that shows that $w \in \operatorname{int} M$. By a), this implies that $f(U) \setminus f(V)$ do not intersect ∂M.

Thus $f(V) = f(U) \cap \partial M$. Since $f(U)$ is open in M, this implies that $f(V)$ is open in ∂M.

We have shown that the restriction $f| \colon V \to f(V)$ is a homeomorphism from an open subset of $\mathbb{R}^{n-1}$ to $f(V)$, which is an open neighbourhood of x in ∂M. This proves that ∂M is $(n-1)$-dimensional manifold and all its points belong to its manifold-interior. In other words the boundary ∂M is an $(n-1)$-manifold without boundary.

3. a) Suppose V is path-connected open subset of $\mathbb{R}^n$, $n \geq 2$, $x \in V$. Prove that $V \setminus \{x\}$ is path-connected by calculating $H_0(V \setminus \{x\})$ (or $\tilde{H}_0(V \setminus \{x\})$).
b) Prove the Jordan-Brouwer separation theorem in $\mathbb{R}^n$, $n \geq 2$: Suppose $B \subset \mathbb{R}^n$ is homeomorphic to S^{n-1}. Then $\mathbb{R}^n \setminus B$ has exactly two path-components U and V, which are both open in $\mathbb{R}^n$. Moreover $\partial U = B = \partial V$, where boundary is taken with respect to $\mathbb{R}^n$.

Solution: a) Consider a portion of long exact reduced homology sequence of the pair $(V, V \setminus \{x\})$,

$$H_1(V, V \setminus \{x\}) \xrightarrow{\Delta} \tilde{H}_0(V \setminus \{x\}) \longrightarrow \tilde{H}_0(V) \tag{0.1}$$

Here $\tilde{H}_0(V) = 0$, since V is path-connected (Corollary 12.6). Since V is open, by exercise 11.1(a)

$$H_1(V, V \setminus \{x\}) \cong H_1(\overline{B}^n, \overline{B} \setminus \{0\}).$$

By long exact reduced homology sequence of the pair $(\overline{B}^n, \overline{B}^n \setminus \{0\})$ (where $\overline{B}^n$ is contractible) and the fact that $\overline{B}^n \setminus \{0\}$ has the same homology groups as S^{n-1} (they are of the same homotopy type) we have that

$$H_1(\overline{B}^n, \overline{B} \setminus \{0\}) \cong \tilde{H}_0(\overline{B} \setminus \{0\}) \cong \tilde{H}_0(S^{n-1}) = 0.$$

Here in the last conclusion it is important to have $n \geq 2$, because for $n = 1$ the space S^0 is not path-connected.

Hence $H_1(V, V \setminus \{x\}) = 0$. Thus both end groups in exact sequence (0.1) are trivial, so, by exactness, $\tilde{H}_0(V \setminus \{x\}) = 0$. By Corollary 12.6. the space $V \setminus \{x\}$ is path-connected.

If we use ordinary long exact homology sequence

$$H_1(V, V \setminus \{x\}) \xrightarrow{\Delta} H_0(V \setminus \{x\}) \xrightarrow{i_*} H_0(V) \tag{0.2}$$

we can no longer say that $H_0(V) = 0$, because it is not, it is (isomorphic to) $\mathbb{Z}$. But the group $H_1(V, V \setminus \{x\})$ is still trivial, so, by exactness, i_* is injection, so $H_0(V \setminus \{x\})$ is (isomorphic to) a subgroup of $\mathbb{Z}$. All (non trivial) subgroups of $\mathbb{Z}$ are isomorphic to $\mathbb{Z}$, so $H_0(V \setminus \{x\}) \cong \mathbb{Z}$, which, by Corollary 12.5., implies that $V \setminus \{x\}$ is path-connected.

b) $\mathbb{R}^n$ is homeomorphic to $S^n \setminus \{x\}$, for any $x \in \mathbb{R}^n$ (example 3.8). This means that
1) we can always regard $\mathbb{R}^n$ as an open subset of S^n and regard any problem in $\mathbb{R}^n$ as a problem in an open subset of S^n.
1) any proper open subset V of S^n is (homeomorphic to) an open subset of $\mathbb{R}^n$.

Suppose $f\colon S^{n-1} \to \mathbb{R}^n$ is a homeomorphism to the image $f(S^{n-1}) = B$. By thinking $\mathbb{R}^n = S^n \setminus \{x\}$, we get $B \cong S^{n-1}$ a subset of S^n, so by Jordan-Brouwer Theorem in S^n (Theorem 17.5)

$$S^n \setminus B = U \cup V,$$

where union is disjoint, U and V are both open and path-connected and $\partial U = B = \partial V$, where boundary is with respect to S^n. A point $x \in S^n$ which we choose so that $\mathbb{R}^n = S^n \setminus \{x\}$ cannot belong to B, so it belongs to U or V. We may assume that $x \in V$. Then U is a subset of $\mathbb{R}^n$ and taking away a point x as above we obtain

$$\mathbb{R}^n \setminus B = U \cup W,$$

where $W = V \setminus \{x\}$. Then U and W are open (open set minus a point is open) in $\mathbb{R}^n$, we know that U is path-connected and W is path-connected by a). Hence U and W are disjoint path components of $\mathbb{R}^n \setminus B$. It remains to show that $\partial U = B = \partial W$, where boundary is **with respect to** $\mathbb{R}^n$. By $\mathrm{cl}_X(A)$ we denote the closure of subset $A \subset X$ with respect to the space X. Then

$$\partial_X A = \mathrm{cl}_X A \setminus \mathrm{int}_X A,$$

in particular, since U and W are both open in $\mathbb{R}^n$, we see that

$$\partial_{\mathbb{R}^n} U = \mathrm{cl}_{\mathbb{R}^n} U \setminus U,$$

$$\partial_{\mathbb{R}^n} W = \mathrm{cl}_{\mathbb{R}^n} W \setminus W.$$

From general topology it is known that if $X \subset Y$, then $\mathrm{cl}_X A = \mathrm{cl}_Y A \cap X$, so in particular, for every subset A of $\mathbb{R}^n$, we have that

$$\mathrm{cl}_{\mathbb{R}^n} A = \mathrm{cl}_{S^n} A \cap \mathbb{R}^n.$$

By Jordan-Brouwer in S^n we know that $\mathrm{cl}_{S^n} U = U \cup B$ and $\mathrm{cl}_{S^n} V = V \cup B$. Now

$$\partial_{\mathbb{R}^n} U = \mathrm{cl}_{\mathbb{R}^n} U \setminus U = (\mathrm{cl}_{S^n} U \cap \mathbb{R}^n) \setminus U = (U \cup B) \setminus U = B$$

(which is half of what we had to show) and

$$\partial_{\mathbb{R}^n} W = \mathrm{cl}_{\mathbb{R}^n} W \setminus W = (\mathrm{cl}_{S^n} W \cap \mathbb{R}^n) \setminus W.$$

We claim that in S^n

$$\mathrm{cl}_{S^n} W = W \cup B \cup \{x\} = V \cup B.$$

Recall that $W = V \setminus \{x\}$. To prove that $\mathrm{cl}_{S^n} W \subset V \cup B$ it is enough to notice that $V \cup B = S^n \setminus U$ is closed in S^n and $W \subset V \subset V \cup B$, so also $\mathrm{cl}_{S^n} W \subset V \cup B$ (closure is the smallest closed set that contains a given set!). Conversely suppose $y \in V$ or $y \in B$. We need to show that $y \in \mathrm{cl}_{S^n} W$. If $y \in W$, this is trivial. Other possibilities are that $y \in B$ or $y = x$. Suppose $y \in B$. Then $y \neq x$. Let A be an arbitrary open neighbourhood of y. Then $A \setminus \{x\}$ is also a neighbourhood of y, hence, since $\partial_{S^n} V = B$, contains a point in V. This point cannot be x, so neighbourhood A intersects $V \setminus \{x\} = W$. We have shown that arbitrary neighbourhood of y intersects W, which is what we wanted. The remaining case is $y = x$. But $x \in V$ so all "small enough " neighbourhoods (in S^n) of x contain in V. Since $\{x\}$ is not open in S^n, all neighbourhoods of x must intersect $V \setminus \{x\} = W$. Thus also $x \in \mathrm{cl}_{S^n} W$.

We have shown that

$$\mathrm{cl}_{S^n} W = W \cup B \cup \{x\},$$

so now we can return to the calculation of $\partial_{\mathbb{R}^n} W$ and finnish it. We have that

$$\partial_{\mathbb{R}^n} W = (\mathrm{cl}_{S^n} W \cap \mathbb{R}^n) \setminus W,$$

where

$$\mathrm{cl}_{S^n} W \cap \mathbb{R}^n = (W \cup B \cup \{x\}) \cup \mathbb{R}^n = W \cup B,$$

and the last union is disjoint. Hence $\partial_{\mathbb{R}^n} W = (W \cup B) \setminus W = B$ and we are done.

4. Provide the details and missing arguments in the following sketch of the original proof Brouwer presented for his fixed point theorem.

Suppose $f\colon \overline{B}^n \to \overline{B}^n$ is continuous and let B_+ and B_- be, as usual, upper and lower (closed) hemispheres of S^n. Using the fact that both B_+ and B_- are homeomorphic to $\overline{B}^n$ we construct a continuous mapping $g\colon S^n \to S^n$ that sends both B_+ and B_- to B_- via f (up to homeomorphisms mentioned above). If f do not have fixed points, $\deg g$ must be $(-1)^{n+1}$. For some reason(?) this is a contradiction.

Solution: The idea is the following. We start by choosing homeomorphisms $f_1\colon B_+ \to \overline{B}^n$ and $f_2\colon B_- \to \overline{B}^n$. Then we define mapping $g\colon S^n \to S^n$ by

$$g(x) = \begin{cases} f_2^{-1} \circ f \circ f_1(x), & \text{if } x \in B_+, \\ f_2^{-1} \circ f \circ f_2(x), & \text{if } x \in B_-. \end{cases}$$

The immediate problem after that is the question is g well-defined i.e. do both formulas give the same result on $B_+ \cap B_- = S^{n-1}$?. If f_1, f_2 are chosen randomly as just some homeomorphisms, g will not be well-defined. Obviously, from the definition of g it is clear that it will be well-defined it $f_1(x) = f_2(x)$ for all $x \in S^{n-1}$. Because of that we proceed as follows. Let $i\colon S^n \to S^n$ be the mapping defined by $i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, -x_{n+1})$. This mapping is inverse of each other, in particular a homeomorphism. Moreover $i(B_-) = B_+$, so i restricts to a homeomorphism $i\colon B_- \to B_+$. Thus first we choose any homeomorphism $f_1\colon B_+ \to \overline{B}^n$. For example simple projection $f_1(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n)$ will do, its inverse is mapping

$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, \sqrt{1 - \sum_{i=1}^n x_i^2})$. Then, as f_2 we define the mapping $f_2 = f_1 \circ i \colon B_- \to \overline{B}^n$. As a composition of homemorphisms it is a homeomorphism. Moreover, $i(x) = x$ for all $x \in S^{n-1}$, so $f_1(x) = f_2(x)$ for all $x \in S^{n-1}$. Thus if we know define $g \colon S^n \to S^n$ by

$$g(x) = \begin{cases} f_2^{-1} \circ f \circ f_1(x), & \text{if } x \in B_+, \\ f_2^{-1} \circ f \circ f_2(x), & \text{if } x \in B_-. \end{cases}$$

using these f_1 and f_2, that will give us well-defined continuous mapping $g \colon S^n \to S^n$. By construction g maps everything to the lower hemisphere B_-, so in particular g is not a surjection. This immediately implies that $\deg g = 0$ (Proposition 18.2.4).

Suppose f did not have any fixed points. Then g also do not have fixed points - this follows from the definition of g. Indeed if $x \notin B_-$, then $g(x) \in B_-$ so equation $g(x) = x$ is impossible. On the other hand if $x \in B_-$ and $g(x) = x$, then $f_2^{-1} \circ f \circ f_2(x) = x$, which implies that $f(f_2(x)) = f_2(x)$, so $f_2(x)$ is a fixed point of f, which contradicts our assumptions.

Thus g do not have fixed points. This means that $g(x) \neq \mathrm{id}(x)$, which can also be written as $g(x) \neq -(h(x))$, for all $x \in S^n$. Here $h \colon S^n \to S^n$ is an antipodal mapping $h(x) = -x$. By Lemma 18.3. g and h are homotopic, so, by Proposition 18.2.3/6 we have that $\deg g = \deg h = (-1)^{n+1}$. This is impossible since we already know that $\deg g = 0$.

Actually we did not even have to know such a fancy fact that $\deg h = (-1)^{n+1}$. It is enough to notice that since h is a homeomorphism its degree must be ± 1, which still cannot be zero.

5. Suppose $f \colon S^n \to S^n$ is an even mapping i.e. such that $f(x) = f(-x)$ for all $x \in S^n$.
   Prove that $\deg f$ is an even integer. Moreover, if n is even, $\deg f = 0$.

   Solution: Since $f(x) = f(-x)$ for all $x \in S^n$ we can factor f through the projective plane $\mathbb{R}P^n$, since it is defined as a quotient space $S^n / \sim$, where $\sim$ is generated by relations of the form $x \sim -x$, $x \in S^n$. Hence there exists $\bar{f} \colon \mathbb{R}P^n \to S^n$ such that $\bar{f} \circ p = f$ (i.e. $\bar{f}(\bar{x}) = f(x)$ for all $x \in S^n$). Here $p \colon S^n \to \mathbb{R}P^n$ is a quotient projection mapping. By the characteristic property of quotient mappings (Lemma 6.2.) $\bar{f}$ is continuous. Taking n-th homology we obtain commutative diagram

$$\begin{array}{ccccc} H_n(S^n) & & \xrightarrow{\quad f \quad} & & H_n(S^n) \\ & \searrow^{p} & & \nearrow^{\bar{f}} & \\ & & H_n(\mathbb{R}P^n) & & \end{array}$$

Now let us pay our attention to the result of exercise 16.10, which tells us precisely what group $H_n(\mathbb{R}P^n)$ and mapping $p\colon H_n(\mathbb{R}P^n) \to H_n(S^n)$ are. We have the following facts:
1) If n is even the group $H_n(\mathbb{R}P^n) = 0$ is trivial. In this case the diagram above implies that $f_* = \bar{f}_* \circ p_*$ must also be zero.
2) If n is odd the group $H_n(\mathbb{R}P^n) \cong \mathbb{Z}$. Moreover we can choose generators α of $H_n(S^n)$ and β of $H_n(\mathbb{R}P^n)$such that $p_*(\alpha) = 2\beta$. In this case

$$f_*(\alpha) = \bar{f}_* \circ p_*(\alpha) = \bar{f}_*(2\beta) = 2\bar{f}_*(\beta).$$

Since $\bar{f}_*(\beta)$ belongs to the group $H_n(S^n)$, it is of the form $m\alpha$ for some $m \in \mathbb{Z}$. It follows that in this case

$$f_*(\alpha) = 2m\alpha,$$

so $\deg f = 2m$ is even.

6. a) Suppose U, V are open and path-connected subsets of $\mathbb{R}^n$ such that $U \cup V = \mathbb{R}^n$. Prove that $U \cap V$ is path-connected (using homology).
b) Have a cup of coffee (or a doughnut) and reflect for a moment would it be easy to prove the claim of a) "elementary", without algebraic topology.
c) Take a moment to appreciate the awesomeness of homology.

Solution: a) Since U and V are open in $\mathbb{R}^n$ and their union is $\mathbb{R}^n$, it follows that $(\mathbb{R}^n; U, V)$ is a proper triad, so there exists (a portion of) exact reduced Mayer-Vietoris sequence

$$\tilde{H}_1(\mathbb{R}^n) \longrightarrow \tilde{H}_0(U \cap V) \longrightarrow \tilde{H}_0(U) \oplus \tilde{H}_0(V)$$

In this sequence $\tilde{H}_1(\mathbb{R}^n) = 0$, since $\mathbb{R}^n$ is contractible, and $\tilde{H}_0(U) \oplus \tilde{H}_0(V) = 0$ since U and V are assumed path-connected. Hence, by exactness $\tilde{H}_0(U \cap V) = 0$, so $U \cap V$ is path-connected.

Strictly speaking the proof above only works if $U \cap V \neq \emptyset$ (since reduced groups are not defined for empty space), but this special case is trivial - empty space is path-connected. In fact intersection $U \cap V$ cannot be empty anyway, since that would contradict the connectedness of $\mathbb{R}^n$.

b)/c) It seems that it would be quite hard to prove the claim directly using topology. Take $x, y \in U \cap V$. In order to prove that $U \cap V$ is path connected we need to construct a path $f\colon I \to U \cap V$ that joins x and y. We know that such a path exists in U or in V but how do we choose the one that lies in both? There may be an "elementary" proof, but it must be quite difficult. The homological proof we presented in a) on the other hand is ridiculously simple. Also, it allows all sorts of generalizations. For example U and V need not to be open, only form "proper triad" w.r.t. $\mathbb{R}^n$. Also the space need not to be $\mathbb{R}^n$ - it is enough to have a space X instead of $\mathbb{R}^n$ with the property $H_1(X) = 0$. Actually it would be enough to have that the mapping $\Delta\colon H_1(X) \to \tilde{H}_0(U \cap V)$ is trivial. Hence the same property is true for example for $X = S^n$ when $n \geq 2$. It is not true for $X = S^1$, however, simple counter-example would be $U = S^1 \setminus \{(0,1)\}$, $V = S^1 \setminus \{(0,-1)\}$. Hence, it also follows that if we want to come up with "elementary" topological proof, we need to use some special properties of the space, which are hard to formulate without using singular homology.

This exercise serves to show how powerful homological methods can be.