Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013
Exercise session 13 (Last) (for the exercise session Tuesday 10.12)

1. Suppose M is an m-manifold and N is an n-manifold.
a) Suppose $m = n$ and M has no boundary. Prove that any continuous injection $f\colon M \to N$ is an open embedding, i.e. is a homeomorphism to its image $f(M)$, which is open in N.
b) Suppose $m > n$. Prove that there are no continuous injections $f\colon M \to N$.

2. Suppose M is an n-manifold.
a) Prove that boundary ∂M and interior $\operatorname{int} M$ are disjoint.
b) Prove that the interior $\operatorname{int} M$ is open in M and itself is an n-manifold without boundary.
c) Prove that the boundary ∂M is closed in M and is an $(n-1)$-manifold without boundary.

3. a) Suppose V is path-connected open subset of $\mathbb{R}^n$, $n \geq 2$, $x \in V$. Prove that $V \setminus \{x\}$ is path-connected by calculating $H_0(V \setminus \{x\})$ (or $\tilde{H}_0(V \setminus \{x\})$).
b) Prove the Jordan-Brouwer separation theorem in $\mathbb{R}^n$, $n \geq 2$: Suppose $B \subset \mathbb{R}^n$ is homeomorphic to S^{n-1}. Then $\mathbb{R}^n \setminus B$ has exactly two path-components U and V, which are both open in $\mathbb{R}^n$. Moreover $\partial U = B = \partial V$, where boundary is taken with respect to $\mathbb{R}^n$.

4. Provide the details and missing arguments in the following sketch of the original proof Brouwer presented for his fixed point theorem.

   Suppose $f\colon \overline{B}^n \to \overline{B}^n$ is continuous and let B_+ and B_- be, as usual, upper and lower (closed) hemispheres of S^n. Using the fact that both B_+ and B_- are homeomorphic to $\overline{B}^n$ we construct a continuous mapping $g\colon S^n \to S^n$ that sends both B_+ and B_- to B_- via f (up to homeomorphisms mentioned above). If f do not have fixed points, $\deg g$ must be $(-1)^{n+1}$. For some reason(?) this is a contradiction.

5. Suppose $f\colon S^n \to S^n$ is an even mapping i.e. such that $f(x) = f(-x)$ for all $x \in S^n$.
Prove that $\deg f$ is an even integer. Moreover, if n is even, $\deg f = 0$. (Fat hint: you are allowed to use the results obtained in the exercise 16.10, in particular the information about $H_n(\mathbb{R}P^n)$ and the mapping $p_*\colon H_n(S^n) \to H_n(\mathbb{R}P^n)$ induced by projection $S^n \to \mathbb{R}P^n$.)

6. [1] a) Suppose U, V are open and path-connected subsets of $\mathbb{R}^n$ such that $U \cup V = \mathbb{R}^n$. Prove that $U \cap V$ is path-connected (using homology).
b) Have a cup of coffee (or a doughnut) and reflect for a moment would it be easy to prove the claim of a) "elementary", without algebraic topology.
c) Take a moment to appreciate the awesomeness of homology.

Bonus points for the exercises: 25% - 2 point, 40% - 3 points, 50% - 4 points, 60% - 5 points, 75% - 6 points.

[1] Suggested by Rami Luisto