# Department of Mathematics and Statistics
# Introduction to Algebraic topology, fall 2013
### Exerciss 11 Solutions

1. a) Suppose $U$ is an open subset of $\mathbb{R}^n$ and let $x \in U$. Use excision to prove that
$$H_k(U, U \setminus \{x\}) \cong H_k(\overline{B}^n, \overline{B}^n \setminus \{0\})$$
for all $k \in \mathbb{Z}$.
b) Suppose $U \neq \emptyset$ is an open subset of $\mathbb{R}^n$, $V$ is an open subset of $\mathbb{R}^m$ and suppose there exists a homeomorphism $f\colon U \to V$. Use a) to prove that $n = m$.

**Solution:** a)Since $U$ is open there exists small enough $r > 0$ such that $\overline{B}(x,r) \subset U$. We wish to excise away a set $A = U \setminus \overline{B}(x,r)$ from the pair $(U, U \setminus \{x\})$. Since
$$A \subset U \setminus B(x,r),$$
where $U \setminus B(x,r)$ is closed in $U$ (since open ball $B(x,r)$ is certainly open), we have that
$$\overline{A} \subset U \setminus B(x,r) \subset U \setminus \{x\} = \operatorname{int} U \setminus \{x\}.$$
Here the last equation follows from the fact that $U \setminus \{x\}$ is open in $U$. By excision theorem 14.1.
$$H_k(U, U \setminus \{x\}) \cong H_k(U \setminus A, (U \setminus \{x\}) \setminus A) = H_k(\overline{B}(x,r), \overline{B}(x,r) \setminus \{x\}),$$
for all $k \in \mathbb{Z}$. Since the pair $(\overline{B}(x,r), \overline{B}(x,r) \setminus \{x\})$ is clearly homeomorphic to the pair $(\overline{B}^n, \overline{B}^n \setminus \{0\})$, we are done.

b) Let $f\colon U \to V$ be a homeomorphism. Since $U$ is not empty we can choose $x \in U$. We can think of $f$ as a mapping of pairs $f\colon (U, U \setminus \{x\}) \to (V, V \setminus \{f(x)\})$. This mappings of pairs is a homeomorphism of pairs. Indeed $f^{-1}\colon V \to U$ exists and is continuous. Moreover it maps $V \setminus \{f(x)\}$ to $U \setminus \{x\}$. Thus $f^{-1}$ can be thought of as a mapping of pairs $f^{-1}\colon (V, V \setminus \{f(x)\}) \to (U, U \setminus \{x\})$, which is then inverse of $f$, thought of as a mapping of pairs. It follows that
$$f_*\colon H_k(U, U \setminus \{x\}) \to H_k(V, V \setminus \{f(x)\})$$
is an isomorphism for all $k \in \mathbb{Z}$ (Corollary 10.13).

On the other hand, by a) $H_k(U, U \setminus \{x\})$ is isomorphic to $H_k(\overline{B}^n, \overline{B}^n \setminus \{0\})$ and likewise $H_k(V, V \setminus \{f(x)\})$ is isomorphic to $H_k(\overline{B}^m, \overline{B}^m \setminus \{0\})$, for all $k \in \mathbb{Z}$. By Exercise 4.b) $H_k(\overline{B}^n, \overline{B}^n \setminus \{0\})$ is isomorphic to $H_k(\overline{B}^n, S^{n-1})$ and likewise $H_k(\overline{B}^m, \overline{B}^m \setminus \{0\})$ is isomorphic to $H_k(\overline{B}^n, S^{m-1})$, for all $k \in \mathbb{Z}$. Using reduced homology sequence of the pair $(\overline{B}^n, S^{n-1})$ and the fact that $\overline{B}^n$ is contractible, we obtain, in the usual manner, that

$$H_k(\overline{B}^n, S^{n-1}) \cong \tilde{H}_{k-1}(S^{n-1}) \cong \begin{cases} \mathbb{Z}, & k = n, \\ 0, & \text{otherwise} \end{cases} .$$

Here the last assertion follows from Theorem 14.2. Similarly

$$H_k(\overline{B}^m, S^{m-1}) \cong \tilde{H}_{k-1}(S^{m-1}) \cong \begin{cases} \mathbb{Z}, & k = m, \\ 0, & \text{otherwise} \end{cases} .$$

Comparing this groups for instance for $k = n$, we see that they can be isomorphic for all $k \in \mathbb{Z}$ if and only if $n = m$.

2. Suppose $f\colon \overline{B}^n \to \overline{B}^n$ is a homeomorphism. Prove that $f(B^n) = B^n$ and $f(S^{n-1}) = S^{n-1}$ (advice: remove a point).

**Solution:** Suppose $\mathbf{x} \in B^n$. We prove that $f(\mathbf{x}) \in B^n$ by making counter-assumption - $f(\mathbf{x}) \in S^{n-1}$. Since $f$ is a homeomorphism, the restriction $f|\colon \overline{B}^n \setminus \{\mathbf{x}\} \to \overline{B}^n \setminus \{f(\mathbf{x})\}$ is also a homeomorphism. We'll derive contradiction by showing that the spaces $\overline{B}^n \setminus \{\mathbf{x}\}$ and $\overline{B}^n \setminus \{f(\mathbf{x})\}$ cannot be homeomorphic. Indeed, since $\mathbf{x} \in B^n$, $\overline{B}^n \setminus \{\mathbf{x}\}$ is homeomorphic to $\overline{B}^n \setminus \{\mathbf{0}\}$ (precise proof of this claim can be obtained through the proof of Theorem 3.20). The latter space is known to have the homotopy type of $S^{n-1}$ (Exercise 5.7.2). Thus it has non-trivial reduced homology group in dimension $n-1$. On the other hand the space $\overline{B}^n \setminus \{f(\mathbf{x})\}$ is contractible. This is seen by noticing that the simple linear homotopy $(x, t) \to tx$ that contracts $\overline{B}^n$ to origin restricts to a contracting homotopy $\overline{B}^n \setminus \{f(\mathbf{x})\} \times I \to \overline{B}^n \setminus \{f(\mathbf{x})\}$, since $f(\mathbf{x})$ is on the boundary. In fact, using the fact that $f(\mathbf{x})$ is an extreme point of the ball $\overline{B}^n$ (which we proved in the exercise 2.4.), we can easily see that $\overline{B}^n \setminus \{f(\mathbf{x})\}$ is even convex (how?). In any case, all its reduced homology groups are trivial. Since we already noticed that $\overline{B}^n \setminus \{\mathbf{x}\}$ has one non-trivial reduced homology groups, the spaces $\overline{B}^n \setminus \{\mathbf{x}\}$ and $\overline{B}^n \setminus \{f(\mathbf{x})\}$ cannot be homeomorphic. This shows that the counter assumption was wrong and $f(\mathbf{x}) \in B^n$.

We have shown that for any homeomorphism $f\colon \overline{B}^n \to \overline{B}^n$ we have that $f(B^n) \subset B^n$. Applying this to $f^{-1}\colon \overline{B}^n \to \overline{B}^n$ we see that

$$B^n = f(f^{-1}(B^n)) \subset f(B^n).$$

Thus $f(B^n) = B^n$. Since $\overline{B^n}$ is a disjoint union of $B^n$ and its boundary $S^{n-1}$ and $f$ is bijection, it follows that we also must have $f(S^{n-1}) \to S^{n-1}$.

3. Suppose $X$ is a non-empty set. We define the chain complex $CX$ by asserting $CX_n$ to be a free abelian group generated by the cartesian product $X^{n+1}$, $n \geq 0$ and $X_n = 0$ for $n < 0$. The boundary operator $d_n\colon CX_n \to CX_{n-1}$, $n \geq 1$, are defined as the unique homomorphism with the property

$$d_n(x_0, \dots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \dots, \hat{x}_i, \dots, x_n)$$

for basis elements $(x_0, \dots, x_n) \in X^{n+1}$ (and $d_n = 0$ for $n < 1$). You do not have to prove that $CX$ is a chain complex (but you are certainly welcome to think about it).
Complex $CX$ has a natural augmentation $\varepsilon\colon CX_0 \to \mathbb{Z}$ defined by $\varepsilon(x) = 1$ for basis elements $x \in X$.

Let $a \in X$ be a fixed element. We define, for every $n \in \mathbb{N}$, a homomorphism $B\colon CX_n(D) \to CX_{n+1}(D)$ by

$$B(x_0, \dots, x_n) = (a, x_0, \dots, x_n)$$

on the basis elements. For $n < 0$ we define $B\colon CX_n(D) \to CX_{n+1}(D)$ to be an obvious zero mapping. Prove that for all $z \in CX_n$ the equation

$$(d_{n+1}B + Bd_n)(z) = \begin{cases} z, & \text{if } n \neq 0, \\ z - \varepsilon(z)a, & \text{if } n = 0 \end{cases}$$

is true.

**Solution:** It is enough to prove that

$$(d_{n+1}B + Bd_n)(z) = \begin{cases} z, & \text{if } n \neq 0, \\ z - \varepsilon(z)a, & \text{if } n = 0 \end{cases}$$

holds when $z = (x_0, \dots, x_n)$ is a generator of $CX_n$. This is a straightforward calculation. We have that

$$(d_{n+1}B(x_0, \dots, x_n) = d_{n+1}(a, x_0, \dots, x_n).$$

To make calculation of boundary operator more clear, let us denote $x'_0 = a$, $x'_i = x_{i-1}$ for $i = 1, \dots, n+1)$. Then

$$d_{n+1}(a, x_0, \dots, x_n) = d_{n+1}(x'_0, \dots, x'_{n+1}) = \sum_{i=0}^{n+1}(-1)^i(x'_0, \dots, \widehat{x'_i}, \dots, x'_{n+1}) =$$

$$= (\hat{a}, x_0, \dots, x_n) + \sum_{i=1}^{n+1}(-1)^i(a, \dots, \widehat{x'_{i-1}}, \dots, x_n).$$

The first element of this sum is simply $z = (x_0, \dots, x_n)$ and shifting index $i$ to $i-1$ in the second sum, we obtain

$$d_{n+1}B(x_0, \dots, x_n) = z + \sum_{i=0}^{n}(-1)^{i+1}B(x_0, \dots, \widehat{x_i}, \dots, x_n) =$$

$$= z - \sum_{i=0}^{n}(-1)^i B(x_0, \dots, \widehat{x_i}, \dots, x_n).$$

Since $B$ is a homomorphism,

$$\sum_{i=0}^{n}(-1)^i B(x_0, \dots, \widehat{x_i}, \dots, x_n) = B(\sum_{i=0}^{n}(-1)^i(x_0, \dots, \widehat{x_i}, \dots, x_n) = B(d_n(z)).$$

Thus

$$(d_{n+1}B + Bd_n)(z) = z - Bd_n(z),$$

which, in case $n > 0$, is what we wanted to prove (for $n < 0$ both sides of equation trivially zero). What happens when $n = 0$? Careful examination of the proof above shows that when $n = 0$, some elements of the proof do not make sense. Namely, when $n = 0$, the "sequence" $(x_0, \dots, \widehat{x_i}, \dots, x_n)$ is empty - the only element is taken away. In this case we cannot say that

$$(a, \dots, \widehat{x_i}, \dots, x_n) = B(x_0, \dots, \widehat{x_i}, \dots, x_n),$$

since it does not hold any more, on the left side we have an element $(a)$ of $CX_0$, which is not in the image of $B_{-1}\colon CX_{-1} \to CX_0$ (later

mapping is trivial).
In case $n = 0$ the calculation looks as following:

$$(d_1 B + B d_0)(z) = d_1(a, x_0) + B(0) = x_0 - a = z - \varepsilon(z)a,$$

since for basis element $z = x_0$ we have that $\varepsilon(z) = 1$.

4. Let

$$B_+ = \{x \in S^n \mid x_{n+1} \geq 0\}$$

and

$$B_- = \{x \in S^n \mid x_{n+1} \leq 0\}$$

Prove that the inclusions of pairs $(B_+, S^{n-1}) \to (S^n, B_-)$ and $(B_-, S^{n-1}) \to (S^n, B_+)$ induce isomorphisms in homology for all dimensions.

**Solution:** Let us start by pondering what the set $A$ must be, so that we can excite it from the pair $(S^n, B_-)$, according to excision property, so that homology stays the same. By excision property we must have that

$$\bar{A} \subset \operatorname{int} B_- = \{x_{n+1} < 0\}.$$

Thus if $A$ has this property, by Theorem 14.1. the inclusion $(S^n \setminus A, B_- \setminus A) \to (S^n, B_-)$ induces isomorphisms in homology in all dimensions.

Now, we cannot obtain $(B_+, S^{n-1})$ this way, because this mean that then we must have $A = \{x_{n+1} < 0\}$ but it is not true that $\bar{A} = B_- \subset \operatorname{int} B_-$. So we start by excising something smaller. The simplest non-trivial choice is

$$A = \{-\mathbf{e}_{n+1}\}.$$

By excision Theorem 14.1. the inclusion $j\colon (S^n \setminus \{-\mathbf{e}_{n+1}\}, B_- \setminus \{-\mathbf{e}_{n+1}\}) \to (S^n, B_-)$ induces isomorphisms $j_*\colon H_k(S^n \setminus \{-\mathbf{e}_{n+1}\}, B_- \setminus \{-\mathbf{e}_{n+1}\}) \to H_k(S^n, B_-)$, for all $k \in \mathbb{Z}$.

Next we get from the pair $(S^n \setminus \{-\mathbf{e}_{n+1}\}, B_- \setminus \{-\mathbf{e}_{n+1}\})$ to the pair $(B_+, S^{n-1})$ by using homotopy axiom. It is enough to show that the inclusion $k\colon (B_+, S^{n-1}) \to (S^n \setminus \{-\mathbf{e}_{n+1}\}, B_- \setminus \{-\mathbf{e}_{n+1}\})$ is a homotopy equivalence of pairs. We construct homotopy inverse $l\colon S^n \setminus \{-\mathbf{e}_{n+1}\}, B_- \setminus \{-\mathbf{e}_{n+1}\}) \to (B_+, S^{n-1})$ as following. For all $x \in B_+$ we simply assert $l(x) = x$.

For $x \in B_-$, $x \neq -\mathbf{e}_{n+1}$ we assert $l(x) = y/|y| \in S^{n-1}$ where $y$ is chosen to be the unique point of $\mathbb{R}^n$ that lies on the line

$$\{-(1-s)\mathbf{e}_{n+1} + sx \mid s \in \mathbb{R}\}$$

(this is the geometrical idea behind the proof). More precisely suppose $s \in \mathbb{R}$ and $x \in B_-$. Then

$$-(1-s)\mathbf{e}_{n+1} + sx = (sx_1, \ldots, s(x_{n+1}+1) - 1),$$

so this point is in $\mathbb{R}^n$ if and only if

$$s = \frac{1}{1 + x_{n+1}}.$$

Notice that since we assume that $x \neq -\mathbf{e}_{n+1}$, $s$ is well-defined (no division by zero). The point of $\mathbb{R}^n$ that we obtain is the point

$$y(x) = \frac{1}{1 + x_{n+1}}(x_1, \ldots, x_n, 0).$$

If that point would be origin $\mathbf{0}$, that would mean $x_1 = \ldots = x_n = 0$, which forces ($x$ is a point in $S^n$) $x = \pm\mathbf{e}_{n+1}$, which is impossible, since $x \in B_-$, $x \neq -\mathbf{e}_{n+1}$. Thus $l(x) = y(x)/|y(x)|$ is well-defined and is continuous as a mapping $B_- \setminus \{-\mathbf{e}_{n+1}\} \to S^{n-1}$ (since $x \mapsto y(x)$ clearly is). If $x \in S^{n-1}$ to begin with, then $y(x) = x$, so $l(x) = x$. This is equivalent to $l \circ k = \mathrm{id}_{S^{n-1}}$. Since we have already defined $l$ also on $B_+$, by $l(x) = x$, we must check that both our definitions agree on $B_+ \cap B_-$. But this intersection is precisely $S^{n-1}$, and we have just shown that both definitions give $l(x) = x$ on $S^{n-1}$.

We have thus constructed a mapping $l\colon S^n \setminus \{-\mathbf{e}_{n+1}\} \to B_+$ such that $l \circ k = \mathrm{id}$. Also $l$ maps $B^n_- \setminus \{-\mathbf{e}_{n+1}\}$ to $S^{n-1}$. Thus $j$ can be thought of as a mapping of pairs $l\colon (S^n \setminus \{-\mathbf{e}_{n+1}\}, B_- \setminus \{-\mathbf{e}_{n+1}\}) \to (B_+, S^{n-1})$.

The next step is to show that $l$ is a homotopy inverse of $k$, as a mapping of pairs. Since $l \circ k = \mathrm{id}$, it is enough to show that $k \circ l$ is homotopic to identity of the pair $(S^n \setminus \{-\mathbf{e}_{n+1}\}, B_- \setminus \{-\mathbf{e}_{n+1}\})$, as a mapping of pair.

We define a homotopy $H\colon S^n \setminus \{-\mathbf{e}_{n+1}\} \times I \to S^n \setminus \{-\mathbf{e}_{n+1}\}$ by

$$H(x,t) = \frac{(1-t)x + tl(x)}{(1-t)x + tl(x)}.$$

Notice that for $x \in B_+$ we have that $H(x,t) = x$ for all $t \in I$.

We need to show that $H$ is well-defined i.e. that
1) $(1-t)x + tl(x) \neq \mathbf{0}$ for all $x \in B_- \setminus \{-\mathbf{e}_{n+1}\}$ and all $t \in I$, and
2) $H(x,t) \neq -\mathbf{e}_{n+1}$ for all $(x,t) \in B_- \setminus \{-\mathbf{e}_{n+1}\} \times I$.
Both claims are clear if $x \in B_+$, so it is enough to prove them assuming $x \in B_-$.

First we prove 1). Suppose on contrary that $(1-t)x + tl(x) = \mathbf{0}$ for some $x \in B_- \setminus \{-\mathbf{e}_{n+1}\}$ and some $t \in I$. If $t = 0$ this implies that $x = \mathbf{0}$, which is impossible. Otherwise

$$l(x) = \frac{t-1}{t} x$$

Since $l(x) \in S^{n-1}$, its last coordinate its zero, so also last coordinate $x_{n+1}$ of $x$ must be zero, which implies that $x \in S^{n-1}$ , in which case $l(x) = x$ and the equation $(1-t)x + tl(x) = \mathbf{0}$ becomes equation $x = \mathbf{0}$, which is impossible. Hence 1) is true.

To prove 2) we notice that the equation $H(x,t) = -\mathbf{e}_{n+1}$ can be true if and only if the first $n$ coordinates of $(1-t)x + tl(x)$ are zeros and $t \neq 0$ (since for $t = 0$ we have that $H(x,0) = x$). Since the last coordinate of $l$ is zero by construction, this implies that

$$l(x) = \frac{t-1}{t}(x_0, \ldots, x_n, 0).$$

On the other hand

$$l(x) = y/|y|,$$

where

$$y = \frac{1}{1 + x_{n+1}}(x_1, \ldots, x_n, 0).$$

But the equation

$$\frac{1}{|y|(1 + x_{n+1})}(x_1, \ldots, x_n, 0) = \frac{t-1}{t}(x_0, \ldots, x_n, 0)$$

is impossible, since on the left side the scalar is always strictly positive and on the right side it is strictly negative (and $x_i \neq 0$ for at least one $i = 0, \ldots, n$, since otherwise $x = \pm\mathbf{e}_{n+1}$). This contradiction concludes the proof of 2.

Thus $H$ is well-defined. It is clearly continuous. By construction it is homotopy between identity and $k \circ l$. It remains to show that it is a homotopy of pairs i.e. maps $B_- \setminus \{-\mathbf{e}_{n+1}\} \times I$ into $B_- \setminus \{-\mathbf{e}_{n+1}\}$. This amounts to showing that the last coordinate of $H(x,t)$ is non-positive, for $(x,t) \in B_- \setminus \{-\mathbf{e}_{n+1}\} \times I$. But this last coordinate has the same sign as $(1-t)x$ i.e. $(1-t)x_{n-1}$ and this is non-positive since $x_{n+1}$ is and $1 - t \geq 0$.

We have concluded the proof of the fact that inclusion of pairs $k\colon (B_+, S^{n-1}) \to (S^n \setminus \{-\mathbf{e}_{n+1}\}, B_- \setminus \{-\mathbf{e}_{n+1}\})$. Now for the inclusion $i\colon (B_+, S^{n-1}) \to (S^n, B_-)$ the triangle

$$
\begin{array}{ccc}
(B_+, S^{n-1}) & \xrightarrow{\quad i \quad} & (S^n, B_-) \\
 & {\scriptstyle k}\searrow \qquad \nearrow{\scriptstyle j} & \\
 & (S^n \setminus \{-\mathbf{e}_{n+1}\}, B_- \setminus \{-\mathbf{e}_{n+1}\}) &
\end{array}
$$

commutes (all mappings just inclusions of pairs). Passing to homology we obtain, for every $k \in \mathbb{Z}$, the commutative diagram

$$
\begin{array}{ccc}
H_k(B_+, S^{n-1}) & \xrightarrow{\quad i_* \quad} & H_k(S^n, B_-) \\
 & {\scriptstyle k_*}\searrow \qquad \nearrow{\scriptstyle j_*} & \\
 & H_k(S^n \setminus \{-\mathbf{e}_{n+1}\}, B_- \setminus \{-\mathbf{e}_{n+1}\}) &
\end{array}
.
$$

In other words $i_* = j_* \circ k_*$ and since $j_*$ and $k_*$ are isomorphisms, also $i_*$ is and we are done.

The similar claim about the inclusion $(B_-, S^{n-1}) \to (S^n, B_+)$ is proved in the same way. You can also use the claim already proved and symmetry. Indeed, consider a mapping $\iota\colon S^n \to S^n$, $\iota(x_0, \ldots, x_n, x_{n+1}) = (x_0, \ldots, x_n, -x_{n+1})$. This mapping maps $B_+$ to $B_-$ and vice versa, $B_-$ to $B_+$. It follows that $\iota$ also maps $S^{n-1}$ to itself, in fact its restriction to $S^{n-1}$ is identity. It follows that we can consider $\iota$ to be a mapping of pairs $\iota\colon (S^n, B_-) \to (S^n, B_+)$ or a mapping of pairs $\iota\colon (S^n, B_+) \to (S^n, B_-)$. Its restriction to $B_+$ can be considered as a mapping of pairs $\iota|\colon (B_+, S^{n-1}) \to (B_-, S^{n-1})$. Likewise its restriction to $B_-$ can be considered as a mapping of pairs $\iota|\colon (B_-, S^{n-1}) \to (B_+, S^{n-1})$. All these mappings are homeomorphisms - in fact $\iota \circ \iota = \mathrm{id}$, so $\iota$ is bijection and inverse of itself. It follows that $\iota\colon (S^n, B_+) \to (S^n, B_-)$ is a continuous inverse of $\iota\colon (S^n, B_-) \to (S^n, B_+)$ and $\iota|\colon (B_-, S^{n-1}) \to (B_+, S^{n-1})$ is a continuous inverse of $\iota|\colon (B_+, S^{n-1}) \to (B_-, S^{n-1})$.

The diagram

$$\begin{array}{ccc} (B_+, S^{n-1}) & \xrightarrow{i_1} & (S^n, B_-) \\ \downarrow{\scriptstyle \iota|} & & \downarrow{\scriptstyle \iota} \\ (B_-, S^{n-1}) & \xrightarrow{i_2} & (S^n, B_+) \end{array}$$

commutes. Here $i_1$ and $i_2$ are inclusions. Hence passing to homology we obtain a commutative diagram

$$\begin{array}{ccc} H_k(B_+, S^{n-1}) & \xrightarrow{(i_1)_*} & H_k(S^n, B_-) \\ \downarrow{\scriptstyle \iota|_*} & & \downarrow{\scriptstyle \iota_*} \\ H_k(B_-, S^{n-1}) & \xrightarrow{(i_2)_*} & H_k(S^n, B_+) \end{array}$$

In this diagram vertical mappings are isomorphisms because they are induced by homeomorhisms. Also we have already managed to show that $i_1$ induces isomorphisms in homology. Hence also $(i_2)_*$ is an isomorphism for all $k \in \mathbb{Z}$.

5. In the course of the proof of the excision property we have defined, for every $n \in \mathbb{Z}$, a barycentric subdivision operator $S_n\colon LC_n(D) \to LC_n(D)$ and the mapping $H_n\colon LC_n(D) \to LC_{n+1}(D)$. We have also shown that $S$ is a chain mapping and $H$ is a chain homotopy between identity mapping $\mathrm{id}\colon LC(D) \to LC(D)$ and $S$. Here $D$ is a convex set of a finite-dimensional vector space.
a) Suppose $X$ is a topological space and let $f\colon \Delta_n \to X$ be a singular $n$-simplex in $X$ i.e. a basis element of $C_n(X)$. We define

$$T_n(f) = f_\sharp(S_n(\mathrm{id}_{\Delta_n})),$$

$$G_n(f) = f_\sharp(H_n(\mathrm{id}_{\Delta_n})),$$

where $S_n\colon LC_n(\Delta_n) \to LC_n(\Delta_n)$ and $H_n\colon LC_n(\Delta_n) \to LC_{n+1}(\Delta_n)$ as above. We extend $T_n$ and $G_n$ to unique homomorphisms $C_n(X) \to C_n(X)$ and $C_n(X) \to C_{n+1}(X)$. Prove that for all $n \in \mathbb{Z}$ we have

$$d_{n+1}G_n + G_{n-1}d_n = \mathrm{id} - T_n.$$

b) Let $m \geq 1$. Prove that

$$\sum_{0 \leq i < m} GT^i$$

is a chain homotopy between the chain mappings id and $T^m$.

**Solution:** a) In the lecture material we have shown that for any convex $D$, $n \geq \mathbb{Z}$ and any element $\alpha$ of $LC_n(D)$ the equation

$$d_{n+1} H_n(\alpha) + H_{n-1} d_n(\alpha) = \alpha - S_n(\alpha). \tag{0.1}$$

Now, suppose $f \colon \Delta_n \to X$ is a singular $n$-simplex in $X$ i.e. a generator of $C_n(X)$. Then

$$d_{n+1} G(f) = d_{n+1} f_\sharp(H_n(\mathrm{id}_{\Delta_n})).$$

Since $f_\sharp$ is a chain mapping

$$d_{n+1} f_\sharp(H_n(\mathrm{id}_{\Delta_n})) = f_\sharp(d_{n+1}(H_n(\mathrm{id}_{\Delta_n})),$$

where in the last equation $d_{n+1}$ is a boundary operator $C_{n+1}(\Delta_n) \to C_n(\Delta_n)$. By (0.1)

$$d_{n+1}(H_n(\mathrm{id}_{\Delta_n}) = \mathrm{id}_{\Delta_n} - S_n(\mathrm{id}_{\Delta_n}) - H_{n-1} d_n(\mathrm{id}_{\Delta_n}).$$

If we apply $f_\sharp$ to this, we obtain (combining with already established results)

$$d_{n+1} f_\sharp(H_n(\mathrm{id}_{\Delta_n})) = f_\sharp(\mathrm{id}_{\Delta_n} - S_n(\mathrm{id}_{\Delta_n}) - H_{n-1} d_n(\mathrm{id}_{\Delta_n})) =$$
$$= f_\sharp(\mathrm{id}_{\Delta_n}) - f_\sharp(S_n(\mathrm{id}_{\Delta_n})) - f_\sharp(H_{n-1} d_n(\mathrm{id}_{\Delta_n})).$$

Here

$$f_\sharp(\mathrm{id}_{\Delta_n}) = f \circ \mathrm{id} = f,$$
$$f_\sharp(S_n(\mathrm{id}_{\Delta_n})) = T_n(f),$$

while

$$f_\sharp(H_{n-1} d_n(\mathrm{id}_{\Delta_n})) = f_\sharp(H_{n-1}(\sum_{i=0}^{n} (-1)^i d_n^i(\mathrm{id}_{\Delta_n})) = \sum_{i=0}^{n} (-1)^i f_\sharp(H_{n-1}(d_n^i(\mathrm{id}_{\Delta_n})).$$

Now, by definition $d_n^i(\mathrm{id}_{\Delta_n}) = \mathrm{id}_{\Delta_n} \varepsilon_n^i = \varepsilon_n^i$, which is **affine** mapping $\Delta_{n-1} \colon \Delta_n$. Hence

$$f_\sharp(H_{n-1} d_n(\mathrm{id}_{\Delta_n})) = \sum_{i=0}^{n} (-1)^i f_\sharp(H_{n-1}(\varepsilon_n^i)).$$

So far we have obtained

$$d_{n+1}T_n(f) = f - T_n(f) - \sum_{i=0}^{n}(-1)^i f_\sharp(H_{n-1}(\varepsilon_n^i)),$$

while what we should obtain in the end is

$$d_{n+1}T_n(f) = f - T_n(f) - G_{n-1}d_n(f),$$

where

$$G_{n-1}d_n(f) = G_{n-1}(\sum_{i=0}^{n}(-1)^i f \circ \varepsilon_n^i) = \sum_{i=0}^{n}(-1)^i G_{n-1}(f \circ \varepsilon_n^i).$$

Here $f \circ \varepsilon_n^i \colon \Delta_{n-1} \to X$ is a basis element of $C_{n-1}(X)$, so by the definition of $G_{n-1}$

$$G_{n-1}(f \circ \varepsilon_n^i) = (f \circ \varepsilon_n^i)_\sharp(H_{n-1}(\mathrm{id}_{\Delta_{n-1}})) = f_\sharp((\varepsilon_n^i)_\sharp(H_{n-1}(\mathrm{id}_{\Delta_{n-1}}))).$$

Thus what we have obtained looks like

$$d_{n+1}T_n(f) = f - T_n(f) - \sum_{i=0}^{n}(-1)^i f_\sharp(H_{n-1}(\varepsilon_n^i)),$$

while what we have to obtain looks like

$$d_{n+1}T_n(f) = f - T_n(f) - \sum_{i=0}^{n}(-1)^i (f_\sharp((\varepsilon_n^i)_\sharp(H_{n-1}(\mathrm{id}_{\Delta_{n-1}}))).$$

Thus it remains to prove that

$$H_{n-1}(\varepsilon_n^i) = (\varepsilon_n^i)_\sharp(H_{n-1}(\mathrm{id}_{\Delta_{n-1}}).$$

Since $\varepsilon_n^i = \varepsilon_n^i \circ \mathrm{id}_{\Delta_{n-1}} = (\varepsilon_n^i)_\sharp(\mathrm{id}_{\Delta_{n-1}})$ we can write this as

$$H_{n-1}(\varepsilon_n^i)_\sharp(\mathrm{id}_{\Delta_{n-1}}) = (\varepsilon_n^i)_\sharp(H_{n-1}(\mathrm{id}_{\Delta_{n-1}}).$$

Now it is easy to see what this is all about - we have to show that $H_{n-1}$ and $(\varepsilon_n^i)_\sharp$ commute (or at least, that would be enough)! This makes sense - homotopy $H$ was "universal construction" defined for all possible convex sets, so it sounds very plausible that it should commute with mappings induced by **affine** mappings, since they are "THE mappings" in the world of convex sets (more precisely - they are the ones that preserve convex structure). Thus what we do next is to attempt to prove the following claim.

**Claim:** Suppose $\alpha\colon D \to E$ is an affine mapping between convex sets. Then $\alpha_\sharp\colon LC_n(D) \to LC_n(E)$ is well-defined and commutes with $H$ for all $n \in \mathbb{Z}$. In other words the diagram

$$\begin{array}{ccc} LC_n(D) & \xrightarrow{\alpha_\sharp} & LC_n(E) \\ \downarrow{\scriptstyle H_n} & & \downarrow{\scriptstyle H'_n} \\ LC_{n-1}(D) & \xrightarrow{\alpha_\sharp} & LC_{n-1}(E) \end{array}$$

commute for all $n \in \mathbb{Z}$.

**Proof of the claim:** Suppose $\alpha\colon D \to E$. By definition $LC_n(D)$ is a free subgroup of $C_n(D)$ generated by all those singular simplices $f\colon \Delta_n \to D$ which are affine mappings and likewise for $D'$. Since $\alpha_\sharp(f) = \alpha \circ f\colon \Delta_n \to D'$ is affine (as a composition of affine mappings) , we see that $\alpha_\sharp$ maps all generators of $LC_n(D)$ into $LC_n(D')$. In other words restriction$\alpha_\sharp\colon LC_n(D) \to LC_n(E)$ is well-defined.

Since $H$ is defined by induction on $n$ (and zero for negative $n$), we prove the second part of the claim by induction on $n$. For $n \leq 0$ there is nothing to prove. Suppose claim is true for $n-1$. We need to prove the claim for $n$. By definition

$$H_n(f) = B_f(f - H_{n-1}df)$$

for singular affine simplex $f\colon \Delta_n \to D$ and similarly

$$H'_n(f) = B_f(f - H'_{n-1}df)$$

for singular affine simplex $f\colon \Delta_n \to D'$. By inductive assumption

$$H'_{n-1} \circ \alpha_\sharp = \alpha_\sharp \circ H_{n-1}$$

and we have to show that

$$H'_n \circ \alpha_\sharp = \alpha_\sharp \circ H_n.$$

For every given affine mapping $f\colon \Delta_n \to D$ the composition $\alpha \circ f$ is an affine mapping $\Delta_n \to D'$, so by definition

$$(H'_n \circ \alpha_\sharp)(f) = H'_n(\alpha_\sharp(f)) = H'_n(\alpha \circ f) = B_{\alpha\circ f}((\alpha \circ f) - H'_{n-1}d(\alpha \circ f)).$$

By inductive assumption, and since $\alpha_\sharp$ is a chain mapping, so commutes with boundary, we have that

$$(\alpha\circ f)-H'_{n-1}d(\alpha\circ f) = \alpha_\sharp(f)-H'_{n-1}\alpha_\sharp(df) = \alpha_\sharp(f)-\alpha_\sharp\circ H_{n-1}(df) = \alpha_\sharp(f-H_{n-1}(df)),$$

so

$$(H'_n \circ \alpha_\sharp)(f) = B_{\alpha\circ f}(\alpha_\sharp(f - H_{n-1}(df)).$$

The next step is to investigate the relation between $B$ and $\alpha_\sharp$. Let $g = (x_0, \dots, x_{n-1})$ be the arbitrary generator element of $LC_{n-1}(D)$, thought of as finite string $(x_0, \dots, x_{n-1})$. Rememeber that this just means that $x_i = g(\mathbf{e}_i)$. Now it follows that

$$\alpha_\sharp(g) = (\alpha(x_0), \dots, \alpha(x_{n-1})),$$

so

$$B_{\alpha\circ f}(\alpha_\sharp(g)) = B_{\alpha\circ f}(\alpha(x_0), \dots, \alpha(x_{n-1})) =$$
$$= (\alpha(f(b)), \alpha(x_0), \dots, \alpha(x_{n-1}) = \alpha_\sharp(f(b), x_0, \dots, x_{n-1}) = \alpha_\sharp \circ B_f(g).$$

Here $b$ is the barycentre of $\Delta_{n-1}$. We have shown that for every generator $g$ of $LC_n(D)$

$$B_{\alpha\circ f} \circ \alpha_\sharp(g) = \alpha_\sharp \circ B_f(g).$$

Since this is true for all generators, it is true for all elements $g \in LC_{n-1}(D)$. In particular it is true for $g = f - H_{n-1}(df)$, so

$$(H'_n\circ\alpha_\sharp)(f) = B_{\alpha\circ f}(\alpha_\sharp(f-H_{n-1}(df)) = \alpha_\sharp\circ B_f(f-H_{n-1}(df)) = \alpha_\sharp H_n(f).$$

The claim is proved.

b) This is in fact an application of the previous week exercise 10.6. In that exercise we have shown that if $f, g\colon C \to D$, $k, m\colon D \to D'$, are chain mappings between chain complexes $C, C', D, D'$, $H$ is a chain homotopy between $f$ to $g$ and $H'$ is a chain homotopy between $k$ and $m$, then $k\circ H+H'\circ g$ is a chain homotopy from $k\circ f$ to $m\circ g$. This gives us directly a way to construct homotopies between composite mappings. We have shown that $G$ is a homotopy between id and $T$, which are both chain mappings $C(X) \to C(X)$. Notice that the fact that $T$ is a chain mapping follows directly from the claim of a, i.e. the existence of homotopy $G$ and Lemma 14.9.

Since $G$ is a homotopy between id and $T$, the result of Exercise 10.6. mentioned above imply that $\mathrm{id}\circ G+G\circ T = G+GT$ is a chain homotopy from $\mathrm{id} \circ \mathrm{id} = \mathrm{id}$ and $T \circ T = T^2$. Iterating in the similar manner by induction we obtain that

$$\sum_{0 \le i < m} GT^i$$

is a chain homotopy between id and $T^m$, for all $m \in \mathbb{Z}$.

6. Suppose $A$ is a retract of $X$. Prove that for all $n \in \mathbb{Z}$

$$H_n(X) \cong H_n(A) \oplus H_n(X, A).$$

**Solution:** It is enough to show that there exists short exact sequence of the form

$$0 \longrightarrow H_n(A) \xrightarrow{f} H_n(X) \xrightarrow{g} H_n(X, A) \longrightarrow 0$$

**which splits**. Now, we do know a certain exact sequence that contains a piece looking like this (without trivial groups on ends) - long exact homology sequence of the pair $(X, A)$,

$$\ldots \longrightarrow H_{n+1}(X, A) \xrightarrow{\Delta} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\Delta} H_{n-1}(A) \longrightarrow \ldots.$$

Now, imagine that we can show that $i_*: H_n(A) \to H_n(X)$ is injective **for all** $n \in \mathbb{Z}$. Then, by exactness $\operatorname{Im} \Delta = \operatorname{Ker} i_* = 0$, so $\Delta: H_{n+1}(X, A) \to H_n(A)$ is trivial zero mapping **for all** $n \in \mathbb{Z}$. This implies, again, by exactness, $\operatorname{Im} g_* = \operatorname{Ker} \Delta = H_n(X, A)$, so $g_*: H_n(X) \to H_n(X, A)$ is surjection **for all** $n \in \mathbb{Z}$. Thus, if $i_*$ is injective for all $n \in \mathbb{Z}$, then also $g_*: H_n(X) \to H_n(X, A)$ is surjection for all $n \in \mathbb{Z}$ and we have an exact sequence

$$0 \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \longrightarrow 0$$

It remains to show that when $A$ is a retract of $X$, then $i_*$ is injective and also that this sequence splits. By Lemma 11.16 the last assertion is equivalent to the existence of homomorphism $f': H_n(X) \to H_n(A)$ such that $f' \circ i_* = \mathrm{id}$. But, if such a homomorphism do exist, then $i_*$ is also automatically injective. This is because if $x, y \in H_n(A)$ are such that $i_*(x) = i_*(y)$, then

$$x = \mathrm{id}(x) = f'(i_*(x)) = f'(i_*(y)) = \mathrm{id}(y) = y.$$

This is an example of the general set-theoretical principle - if a mapping has so-called "left inverse" w.r.t. composition of mappings, it must be injective.

We have reduced the problem to finding a homomorphism $f' \colon H_n(X) \to H_n(A)$ such that $f' \circ i_* = \mathrm{id}$. Since $A$ is a retract, there exists continuous mapping $r \colon X \to A$ such that $r(x) = x$ for all $x \in A$. This is the same thing as the equation $r \circ i = \mathrm{id}_A$. Taking stars of both sides we obtain

$$r_* \circ i_* = (r \circ i)_* = \mathrm{id}_* = \mathrm{id}\,.$$

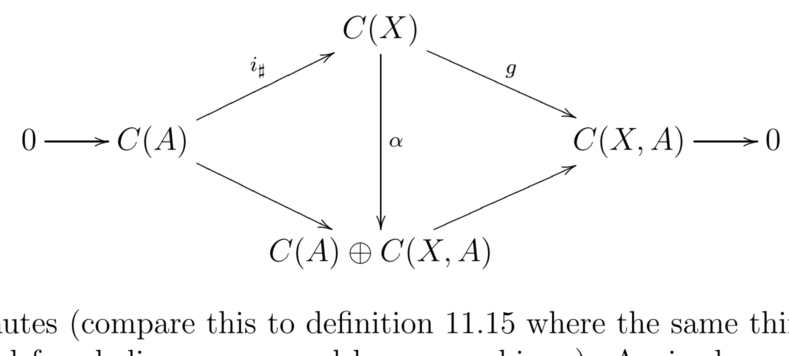
Thus $r_*$ is a mapping we are looking for.

The proof above is "abstract" and "axiomatic" - we did not use actual construction and definition of singular homology, only its properties such as long exact homology sequence or the properties of star-operator. There is also a direct approach. Namely it is enough to prove that $C(X) \cong C(A) \oplus C(X, A)$ **as chain complexes** (be careful - on the group level they are always isomorphic, $C_n(X) \cong C_n(A) \oplus C_n(X, A)$ for all $n \in \mathbb{Z}$ and any pair $(X, A)$, but this is not enough). The claim follows then from Lemma 12.1 (passing to homologies preserve isomorphisms of chain complexes and direct sums of chain complexes).

To prove that $C(X) \cong C(A) \oplus C(X, A)$ as chain complexes, it is enough to show that the exact sequence

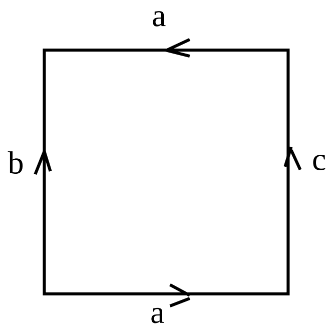
$$0 \longrightarrow C(A) \xrightarrow{i_\sharp} C(X) \xrightarrow{j_\sharp} C(X, A) \longrightarrow 0$$

of chain complexes and chain mappings **splits**. In complete analogy to situation for groups this means that there exists a **chain mapping** $\alpha \colon C(X) \to C(A) \oplus C(X, A)$ such that the diagram




commutes (compare this to definition 11.15 where the same thing was defined for abelian groups and homomorphisms). Again, by complete analogy, the version of Lemma 11.16 works in chain complex settings (you have to check that mappings constructed in the proof of this Lemma sum up to a chain mapping), so it is enough to show the existence of **chain mapping** $f' \colon C(X) \to C(A)$ with the property $f' \circ i_\sharp = \mathrm{id}$. The mapping $r_\sharp$, where $r \colon X \to A$ is a retraction, satisfies this property.

7.* By "the boundary" $dM$ of the Mobius Band we mean the union of sides $b$ and $c$ as a subset of $M$ as in the picture below



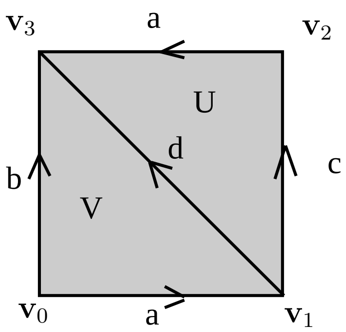

By investigating groups $H_n(M, dM)$ (or other methods) prove the following facts:
a) $dM$ is not a retract of $M$
b) $H_1(M) \cong H_1(dM) \cong \mathbb{Z}$ and it is possible to choose generators in groups $H_1(M)$, $H_1(dM)$ so that the mapping $i_* \colon H_1(dM) \to H_1(M)$ induced by inclusion $i \colon dM \hookrightarrow M$ can be thought of as a mapping $\mathbb{Z} \to \mathbb{Z}$, $n \mapsto 2n$.

**Solution:** a) We can try to apply the precious Lemma, which implies that it would be enough to show that

$$H_n(M) \cong H_n(dM) \oplus H_n(X, dM)$$

is not true for at least one $n \in \mathbb{Z}$. Since the pair $(M, dM)$ can be triangulated as the pair $(K, L)$ of $\Delta$-complexes, all groups can be calculated as simplicial homology (Theorem 15.1).

The simplicial structure of $K$ and $L$ we use is the following familiar structure:




The simplicial homology of Mobius band triangulated as in the picture we calculated in Example 9.6. The results were

$$H_n(K) = \begin{cases} \mathbb{Z}, & \text{for } n = 0, 1 \\ 0, & \text{otherwise .} \end{cases}$$

Let us calculate $H_1(K, L)$. The group $C_2(K, L)$ is a free abelian group of two generators $U$ and $V$ (or rather their classes in the quotient group $C_2(K, L)$, but we use this simpler notation). Since in $K$ we have

$$dU = a + c - d,$$

$$dV = a + d - b,$$

and $b, c \in L$, in $C_1(K, L)$ we have

$$dU = a - d,$$

$$dV = a + d$$

(again we do not bother with denoting equivalence classes as classes, for simplicity of notation). It follows that the boundary group $B_1(K, L)$ is a group generated on $a - d$ and $a + d$.

The group $C_1(K, L)$ is generated on two elements $a, d$. Since all vertices of $K$ belong to $L$, we have that $C_0(K, L) = 0$, so

$$Z_1(K, L) = \operatorname{Ker} d_1 = C_1(K, L) = \mathbb{Z}[a] \oplus \mathbb{Z}[d].$$

Thus we have that

$$H_1(K, L) = \mathbb{Z}[a] \oplus \mathbb{Z}[d]/(\mathbb{Z}[a + d] \oplus \mathbb{Z}[a - d]).$$

To calculate this we do the standard switch of basis-trick, using Exercise 7.2. According to this exercise $\{a + d + (a - d), a - d\} = \{2a, a - d\}$ is a basis of $B_1(K, L)$ and $\{a, a - d\}$ is a basis of $Z_1(K, L)$. Hence

$$H_1(K, L) = \mathbb{Z}[a] \oplus \mathbb{Z}[a - d]/(\mathbb{Z}[2a] \oplus \mathbb{Z}[a - d]) \cong \mathbb{Z}[a]/\mathbb{Z}[2a] \cong \mathbb{Z}_2.$$

This is actually enough - using this information we can already prove that

$$H_1(M) \cong H_1(dM) \oplus H_1(X, dM).$$

Indeed the group on the left is $\mathbb{Z}$, as we know from Example 9.6. The group on the right, on the other hand, contains a summond $H_1(X, dM) \cong \mathbb{Z}_2$, so the group on the right contains at least one non-trivial torsion element, namely $(0, \bar{1})$. Free group cannot contain non-trivial torsion element, so this is a contradiction that proves a).

b) The mapping $i_* \colon H_1(dM) \to H_1(M)$ features in the long exact reduced homology sequence of the pair $(M, dM)$

$$\dots \longrightarrow H_2(M, dM) \xrightarrow{\Delta_2} H_1(dM) \xrightarrow{i_*} H_1(M) \xrightarrow{j_*} H_1(M, dM) \xrightarrow{\Delta_1} \tilde{H}_0(dM) \longrightarrow \dots$$

The group $\tilde{H}_0(dM)$ is trivial, since $dM$ is path-connected (it is actually homeomorphic to $S^1$). To calculate $H_2(M, dM)$ we use simplicial homology again. Actually, using things we have already calculated above in a), we obtain for all $n, m \in \mathbb{Z}$

$$d_2(nU + mV) = n(a + d) - m(a - d) = (n + m)a + (n - m)d.$$

Here $U, V$ are free generators of $C_2(K, L)$ and $a, d$ are free generators of $C_1(K, L)$. Thus $d_2(nU + mV) = 0$ if and only if $n + m = n - m$ which is easily seen to imply $n = m = 0$. Hence $d_2$ is injection, so $Z_2(K, L) = \operatorname{Ker} d_2 = 0$ and consequently $H_2(M, dM$ is trivial.

Thus we obtain short exact sequence

$$0 \longrightarrow H_1(dM) \xrightarrow{i_*} H_1(M) \xrightarrow{j_*} H_1(M, dM) \longrightarrow 0.$$

Here we know that $H_1(M) \cong \mathbb{Z}$ and $H_1(M, dM) \cong \mathbb{Z}_2$. The group $H_1(dM)$ we do not know, but we can calculate for example simplicially again, or we can use the fact that, by exactness, it is (isomorphic to) a subgroup of $H_1(M) \cong \mathbb{Z}$. All subgroups of $\mathbb{Z}$ are either trivial or isomorphic to $\mathbb{Z}$. Trivial $H_1(M)$ cannot be, since then by exactness $j_*$ is isomorhism, which is impossible, since $\mathbb{Z}$ and $\mathbb{Z}_2$ are not isomorphic. Hence $H_1(M) \cong \mathbb{Z}$. Or, one can argue that $dM$ is easily seen to be homeomorphic to $S^1$ and we do know that $H_1(S^1) \cong \mathbb{Z}$.

All in all, short exact sequence

$$0 \longrightarrow H_1(dM) \xrightarrow{i_*} H_1(M) \xrightarrow{j_*} H_1(M, dM) \longrightarrow 0.$$

up to isomorphisms look like short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}_2 \longrightarrow 0.$$

Let us investigate what $\alpha$ and $\beta$ can be. Since $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$, there are only two possibilities, $\beta(1) = \bar{0}$ or $\beta(1) = \bar{1}$. But if $\beta(1) = \bar{0}$, then $\beta(x) = \beta(x \cdot 1) = x\bar{0} = \bar{0}$ for all $x \in \mathbb{Z}$ (since $\beta$ homomorphism), so $\beta$ cannot be surjective, which contradicts exactness (it has to be surjection). Hence $\beta(1) = \bar{1}$, i.e. $\beta\colon \mathbb{Z} \to \mathbb{Z}_2$ is nothing but canonical projection $\beta(x) = \bar{x}$ from $\mathbb{Z}$ to the quotient group $\mathbb{Z} \to \mathbb{Z}_2$. The kernel of such a mapping is $2\mathbb{Z}$, the subgroup of all even integers. By exactness it is the image of $\alpha\colon \mathbb{Z} \to \mathbb{Z}$. Now, an isomorphism $\alpha\colon \mathbb{Z} \to \mathbb{Z}$ is completely determined by $n = \alpha(1)$, in which case $\alpha(x) = nx$ for all $x \in \mathbb{Z}$ and $\operatorname{Im}\alpha = n\mathbb{Z}$. It follows that there are exactly two homomorphisms $\alpha\colon \mathbb{Z} \to \mathbb{Z}$ that fit in the short exact sequence above - one is mapping $x \mapsto 2x$ and the other is $x \mapsto -2x$ (don't forget about the other possibility, it is easy to overlook it!). Since the sequences

$$0 \longrightarrow H_1(dM) \xrightarrow{i_*} H_1(M) \xrightarrow{j_*} H_1(M, dM) \longrightarrow 0.$$

and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}_2 \longrightarrow 0.$$

are isomorphic it follows that either $i_*(u) = 2v$ or $i_*(u) = -2v$, where $u$ is a chosen generator of $H_1(dM)$ and $v$ is a chosen generator of $H_1(M)$. But $-v$ is then also a generator of $H_1(M)$, so re-choosing a generator, we obtain that $i_*(u) = 2v$. This is what had to be shown.

Of course there is more direct way obtain the same result - using singular homology we can compute the actual generators for both $H_1(dM)$ and $H_1(M)$ and use them to calculate $i_*$. Since $C_1(L)$ is generated by two 1-simplices $b$ and $c$ of $L$ and

$$d_1 b = -d_1 c$$

(check!), we see that $Z_1(L) = \mathbb{Z}[b - c]$. Since there are no 2-simplices in $L$, $B_1(L)$ is trivial, so $H_1(L) \cong Z_1(L)$ is a free group generated by the class of $b + c$.
The generator for $H_1(K)$ was calculated in the example 9.6., the result was that as a generator for $H_1(K) \cong \mathbb{Z}$ we can choose the homology class of the cycle $d$ (the diagonal of the square). Now we claim that in $H(K)$

$$[b + c] = 2[d],$$

which would be sufficient for the claim. This can be seen as follows. In $C_1(K)$ the elements

$$-dU = d - a - c,$$

$$dV = d + a - b$$

are boundaries, hence vanish in homology. Adding them together we obtain in homology

$$0 = [d(V - U)] = [d - a - c + d + a - b] = [2d - (b + c)],$$

which is the same as $[b + c] = 2[d]$ and we are done.

**Remark:** Actually it is enough to do b), since a) can be obtained as a simple corollary of b). Indeed, suppose $dM$ is a retract of $M$. This means that there exists continuous mapping $r\colon M \to dM$ such that $r \circ i = \mathrm{id}$. Taking stars of this equations gives us

$$r_* \circ i_* = \mathrm{id}\,.$$

Let $u$ be the generator of $H_1(dM)$. Then $i_*(u) = 2v$ for a generator $v$ of $H_1(M)$. This implies that

$$u = \mathrm{id}(u) = r_*(i_*(u)) = r_*(2v) = 2r_*(v).$$

This equation cannot be true, since $r_*(v) = mu$ for some $m \in \mathbb{Z}$ and equation $u = 2mu$ is impossible. Essentially if you think of the situation in terms of mappings $\mathbb{Z} \to \mathbb{Z}$, then $i_*$ is mapping $x \mapsto 2x$ and such a homomorphism do not have left inverse $r_*\colon \mathbb{Z} \to \mathbb{Z}$, which would be also a homomorphism.