Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013
Exercise session 11 (for the exercise session Tuesday 26.11)

1. a) Suppose U is an open subset of $\mathbb{R}^n$ and let $x \in U$. Use excision to prove that
$$H_k(U, U \setminus \{x\}) \cong H_k(\overline{B}^n, \overline{B}^n \setminus \{0\})$$
for all $k \in \mathbb{Z}$.
b) Suppose $U \neq \emptyset$ is an open subset of $\mathbb{R}^n$, V is an open subset of $\mathbb{R}^m$ and suppose there exists $f\colon U \to V$ is a homeomorphism. Use a) to prove that $n = m$.

2. Suppose $f\colon \overline{B}^n \to \overline{B}^n$ is a homeomorphism. Prove that $f(B^n) = B^n$ and $f(S^{n-1}) = S^{n-1}$ (advice: remove a point).

3. Suppose X is a non-empty set. We define the chain complex CX by asserting CX_n to be a free abelian group generated by the cartesian product X^{n+1}, $n \geq 0$ and $X_n = 0$ for $n < 0$. The boundary operator $d_n\colon CX_n \to CX_{n-1}$, $n \geq 1$, are defined as the unique homomorphism with the property
$$d_n(x_0, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \ldots, \hat{x}_i, \ldots, x_n)$$
for basis elements $(x_0, \ldots, x_n) \in X^{n+1}$ (and $d_n = 0$ for $n < 1$). You do not have to prove that CX is a chain complex (but you are certainly welcome to think about it).
Complex CX has a natural augmentation $\varepsilon\colon CX_0 \to \mathbb{Z}$ defined by $\varepsilon(x) = 1$ for basis elements $x \in X$.

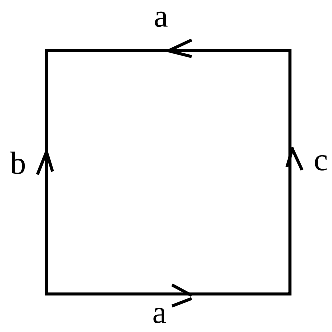
Let $a \in X$ be a fixed element. We define, for every $n \in \mathbb{N}$, a homomorphism $B\colon CX_n(D) \to CX_{n+1}(D)$ by
$$B(x_0, \ldots, x_n) = (a, x_0, \ldots, x_n)$$
on the basis elements. For $n < 0$ we define $B\colon LC_n(D) \to LC_{n+1}(D)$ to be an obvious zero mapping. Prove that for all $z \in CX_n$ the equation
$$(d_{n+1}B + Bd_n)(z) = \begin{cases} z, \text{ if } n \neq 0, \\ z - \varepsilon(z)a, \text{ if } n = 0 \end{cases}$$
is true.

4. Let
$$B_+ = \{x \in S^n \mid x_{n+1} \geq 0\}$$
and
$$B_- = \{x \in S^n \mid x_{n+1} \leq 0\}$$
Prove that the inclusions of pairs $(B_+, S^{n-1}) \to (S^n, B_-)$ and $(B_-, S^{n-1}) \to (S^n, B_+)$ induce isomorphisms in homology for all dimensions.

5. In the course of the proof of the excision property we have defined, for every $n \in \mathbb{Z}$, a barycentric subdivision operator $S_n \colon LC_n(D) \to LC_n(D)$ and the mapping $H_n \colon LC_n(D) \to LC_{n+1}(D)$. We have also shown that S is a chain mapping and H is a chain homotopy between identity mapping $\mathrm{id} \colon LC(D) \to LC(D)$ and S. Here D is a convex set of a finite-dimensional vector space.
a) Suppose X is a topological space and let $f \colon \Delta_n \to X$ be a singular n-simplex in X i.e. a basis element of $C_n(X)$. We define
$$T_n(f) = f_\sharp(S_n(\mathrm{id}_{\Delta_n})),$$
$$G(f) = f_\sharp(H_n(\mathrm{id}_{\Delta_n})),$$
where $S_n \colon LC_n(\Delta_n) \to LC_n(\Delta_n)$ and $H_n \colon LC_n(\Delta_n) \to LC_{n+1}(\Delta_n)$ as above. We extend T_n and G_n to unique homomorphisms $C_n(X) \to C_n(X)$ and $C_n(X) \to C_{n+1}(X)$. Prove that for all $n \in \mathbb{Z}$ we have
$$d_{n+1}G_n + G_{n-1}d_n = \mathrm{id} - T_n.$$
b) Let $m \geq 1$. Prove that
$$\sum_{0 \leq i < m} GT^i$$
is a chain homotopy between the chain mappings id and T^m.

6. Suppose A is a retract of X. Prove that for all $n \in \mathbb{Z}$
$$H_n(X) \cong H_n(A) \oplus H_n(X, A).$$

7.* By "the boundary" dM of the Mobius Band we mean the union of sides b and c as a subset of M as in the picture below



By investigating groups $H_n(M, dM)$ (or other methods) prove the following facts:
a) dM is not a retract of M
b) $H_1(M) \cong H_1(dM) \cong \mathbb{Z}$ and it is possible to choose generators in groups $H_1(M)$, $H_1(dM)$ so that the mapping $i_*\colon H_1(dM) \to H_1(M)$ induced by inclusion $i\colon dM \hookrightarrow M$ can be thought of as a mapping $\mathbb{Z} \to \mathbb{Z}$, $n \mapsto 2n$.
(Hint: simplicial homology)

Bonus points for the exercises: 25% - 2 point, 40% - 3 points, 50% - 4 points, 60% - 5 points, 75% - 6 points.