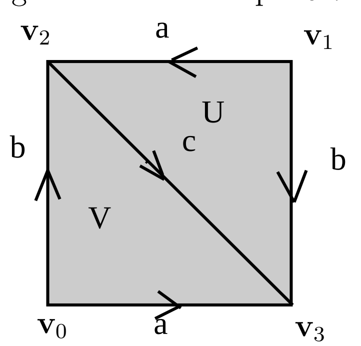
Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013
Exercises 8 (for the exercise session Tuesday 5.11.)

1. a) Suppose $f\colon X \to Y$ is a continuous mapping between topological spaces X and Y. Show that the collection of mappings $f_\sharp\colon C_n(X) \to C_n(Y)$ defined in Example 10.1. is a chain mapping.

   b) Suppose $f\colon C \to D$ is a chain mapping between chain complexes C, D. Suppose $f_n\colon C_n \to D_n$ is a bijection for every $n \in \mathbb{Z}$. Show that f is an isomorphism of chain complexes.

2. Suppose that $f\colon C \to D$ is a chain mapping between chain complexes C, D.
   a) Show that $\operatorname{Ker} f$ is a subcomplex of C, $\operatorname{Im} f$ is a subcomplex of D.
   b) Suppose C' is a subcomplex of C. We denote by $p\colon C \to C/C'$ the canonical projection to the quotient complex. Show that there exists a chain mapping $\bar{f}\colon C/C' \to D$ such that $\bar{f} \circ p = f$ if and only if $C' \subset \operatorname{Ker} f$.

3. Let K be Δ-complex whose polyhedron is the projective plane $\mathbb{R}P^2$, given in the example 9.7.



   Let L be the subcomplex consisting of 1-simplex c and its vertices. Calculate homology groups $H_n(K, L)$ for all $n \in \mathbb{Z}$ directly from definition.

4. Suppose $f\colon C \to D$ is a chain mapping between chain complexes C, D. We define a complex $\bar{C}$ (called the cone of f) as following. For every $n \in \mathbb{Z}$ we assert

$$\bar{C}_n = C_{n-1} \oplus D_n,$$

$$\bar{d}_n(a, b) = (-d_{n-1}(a), f(a) + d'_n(b)).$$

Prove that $\bar{C}$ equipped with boundary operators $\bar{d}_n$ is a chain complex. Is the collection of subgroups

$$C'_n = \{(a, 0) \mid a \in C_n\}, n \in \mathbb{Z}$$

a subcomplex of $\bar{C}$?

5. Suppose $f\colon C \to D$ is a chain mapping between chain complexes C, D and let $\bar{C}$ be a cone of f defined in the previous exercise. We define $j_n\colon D_n \to \bar{C}_n$ by $j_n(b) = (0, b)$ for every $b \in D_n$ and every $n \in \mathbb{Z}$.
   a) Show that j_n is injective for all $n \in \mathbb{Z}$ and that the collection of mappings j_n is a chain mapping $j\colon D \to \bar{C}$.
   b) For every $n \in \mathbb{Z}$ let $p_n\colon \bar{C}_n \to C_{n-1}$ be the mapping defined by $p_n(a, b) = a$. Is the diagram

$$\begin{array}{ccc} \bar{C}_n & \xrightarrow{p_n} & C_{n-1} \\ \downarrow d & & \downarrow d \\ \bar{C}_{n-1} & \xrightarrow{p_{n-1}} & C'_{n-2}. \end{array}$$

   commutative? If not how can it be easily fixed to be commutative?
   c) By a) we can identify D with the subcomplex $j(D)$ of $\bar{C}$. Show that for the quotient complex $\bar{C}/D$ we have for every $n \in \mathbb{Z}$ that

$$(\bar{C}/D)_n \cong C_{n-1} \text{ and}$$

$$H_n(\bar{C}/D) \cong H_{n-1}(C).$$

   Is quotient complex $\bar{C}/D$ isomorphic to the complex C?

6. Suppose A and B are abelian groups. Show that the sequence

$$0 \longrightarrow A \xrightarrow{i} A \oplus B \xrightarrow{q} B \longrightarrow 0,$$

   is a short exact sequence. Here $i\colon A \to A \oplus B$ and $q\colon A \oplus B \to B$ are defined by

$$i(a) = (a, 0)$$

$$q(a, b) = b.$$

Bonus points for the exercises: 25% - 2 point, 40% - 3 points, 50% - 4 points, 60% - 5 points, 75% - 6 points.