Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013

Exercises 6 (for the exercise session Tuesday 15.10)

1. An element x of an abelian group G is called *torsion* element if there exists $n \in \mathbb{Z}, n > 0$ such that $nx = 0$ (where 0 is a neutral element of G). The set of all torsion elements of G is denoted $\operatorname{Tor}(G)$. If $\operatorname{Tor}(G) = \{0\}$, G is called *torsion free*.
   a) Prove that $\operatorname{Tor}(G)$ is always a subgroup of G.
   b) Show that quotient group $G/\operatorname{Tor}(G)$ is always torsion free.
   c) Show that the torsion subgroup $\operatorname{Tor}(\mathbb{R}/\mathbb{Z})$ of the quotient group $\mathbb{R}/\mathbb{Z}$ is $\mathbb{Q}/\mathbb{Z}$.

   In the exercise 2 you need to recall complex numbers and their multiplication. The set $\mathbb{C}$ of complex numbers is $\mathbb{R}^2$ i.e. the set of real-valued pairs (x, y). The multiplication of complex numbers is defined by

$$(x, y) \cdot (u, v) = (xu - yv, xv + yu).$$

   It is assumed known that the set of all non-zero complex numbers $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ is an abelian group when equipped with this multiplication. The neutral element is $1 = (1, 0)$. The set of complex numbers of norm 1

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

   is a subgroup of $\mathbb{C}^*$. All elements of S^1 can be represented in the form

$$(\cos\alpha, \sin\alpha)$$

   where angle α is unique up to a multiply of 2π.

2. For every $n \in \mathbb{N}, n > 0$ we let

$$C_n = \{z \in \mathbb{C} \mid z^n = 1\}.$$

   a) Show that C_n is isomorphic to $\mathbb{Z}_n$ for all $n > 0$.
   b) Show that

$$\operatorname{Tor}\mathbb{C}^* = \bigcup_{n>0} C_n$$

   c) Consider the mapping $f: \mathbb{R} \to \mathbb{C}^*$,

$$f(x) = (\cos 2\pi x, \sin 2\pi x).$$

   Show that $f: (\mathbb{R}, +) \to (\mathbb{C}^*, \cdot)$ is a homomorphism of abelian groups and use f to prove that the quotient groups $\mathbb{R}/\mathbb{Q}$ and $S^1/\operatorname{Tor}\mathbb{C}^*$ are isomorphic.

3. Let A be a set. For every $a \in A$ we define $f_a \colon A \to \mathbb{Z}$ by

$$f_a(x) = \begin{cases} 1, & \text{if } x = a, \\ 0, & \text{otherwise} . \end{cases}$$

   a) Prove that $\mathbb{Z}^A$ is an abelian group (with point-wise addition, see lecture notes) and $\mathbb{Z}^{(A)}$ is its subgroup.
   b) Prove that

$$\{f_a \mid a \in A\}$$

   is a basis of $\mathbb{Z}^{(A)}$.

4. Suppose G is an abelian group and $(H_\alpha)_{\alpha \in \mathcal{A}}$ is indexed collection of abelian groups. Suppose for every $\alpha \in \mathcal{A}$ a homomorphism of abelian groups $f_\alpha \colon G \to H_\alpha$ is given. Prove that there exists unique homomorphism of groups

$$f \colon G \to \prod_{\alpha \in \mathcal{A}} H_\alpha$$

   such that $\mathrm{pr}_\alpha \circ f = f_\alpha$ for all $j \in \mathcal{A}$.

5. Prove that
   a) $\mathbb{Z}_6$ is isomorphic to the direct sum $\mathbb{Z}_2 \oplus \mathbb{Z}_3$,
   b) $\mathbb{Z}_4$ is not isomorphic to the direct sum $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.
   What do you think is the essential difference between those cases responsible for these results? Try to conjecture some general results similar to those special cases (you do not have to prove your conjectures, of course).

6. a) Suppose A is a linearly independent subset of the abelian group $(\mathbb{Q}, +)$. Prove that A has at most one element.
   b) Use a) to prove that $\mathbb{Q}$ is not free.
   c) Show also that $\mathbb{Q}$ is not finitely generated.

Bonus points for the exercises: 25% - 2 point, 40% - 3 points, 50% - 4 points, 60% - 5 points, 75% - 6 points.