Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013
Exercises 5. Solutions

1. Suppose $f\colon X \to Y$ is a quotient mapping and $g\colon Y \to Z$ is a mapping (not assumed to be continuous). Prove that g is continuous if and only if the composition mapping $g \circ f\colon X \to Z$ is continuous. X, Y, Z are topological spaces.

   Solution: The composition of two continuous mappings is continuous, this is well-known fact from the general topology. Hence if g is continuous, then $g \circ f$ is continuous, since any quotient mapping is continuous (follows from the definition)

   Conversely let us assume that $f\colon X \to Y$ is a quotient mapping and $g\colon Y \to Z$ is a mapping, such that $g \circ f\colon X \to Z$ is continuous. By Lemma 3.2 in order to prove that g is continuous, it is enough to show that the inverse image $g^{-1}U$ of an open subset $U \subset Z$ is an open subset of Y.

   Since f is a quotient mapping, $g^{-1}U \subset Y$ is open if and only if $f^{-1}(g^{-1}U)$ is open in X. But this is true, since by set theory we have that
$$f^{-1}(g^{-1}U) = (g \circ f)^{-1}U$$
   and $g \circ f$ is assumed continuous.

2. By $\mathbb{R}P^n$ (*real projective space*) we denote the quotient space $S^n/\sim$ of the sphere S^n with respect to the equivalence relation $\sim$ generated by relations $(x, -x)$, $x \in S^n$.
   a) Prove that the canonical projection $p\colon S^n \to \mathbb{R}P^n$ is both an open and a closed mapping (note: the fact that $\mathbb{R}P^n$ is Hausdorff is not obvious, so if you want to use it, you have to prove it separately).
   b) Consider a mapping $f\colon \overline{B}^n \to S^n$ defined by
$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_n, \sqrt{1 - \sum_{i=1}^{n} x_i^2}).$$
   Show that the composite $p \circ f$ is a quotient mapping (Hint: show that it is a closed surjection).
   c) Use b) to show that $\mathbb{R}P^n$ is homeomorphic to the quotient space

$\overline{B}^n / \sim'$, where $\sim'$ is an equivalence relation on $\overline{B}^n$ generated by the relations $(x, -x)$, $x \in S^{n-1}$ (notice - identifications only on the boundary!).

Solution: a) Let $U \subset S^n$ be open. We need to show that $p(U)$ is open in $\mathbb{R}P^n$. Since $\mathbb{R}P^n$ has the quotient topology and $p\colon S^n \to \mathbb{R}P^n$ is a canonical projection, a subset $p(U)$ of $\mathbb{R}P^n$ is open in $\mathbb{R}P^n$ if and only $p^{-1}p(U)$ is open in S^n. Now, an element $y \in S^n$ belongs to $p^{-1}p(U)$ if and only if

$$p(y) = \bar{y} = \bar{x} = p(x)$$

for some $x \in p(U)$. This means that $y \sim x$ for some $x \in U$, which, by definition of $\sim$ means precisely that $y \in U$ or $-y \in U$ i.e. $y \in U$ or $y \in -U$. Thus

$$p(U) = U \cup (-U).$$

Since the antipodal mapping $\iota\colon S^n \to S^n$, $\iota(x) = -x$ is obviously a homeomorphism (it is continuous bijection, whose inverse is ι itself), we have that $-U = \iota(U)$ is open in S^n. Hence $p(U)$ is open as a union of two open sets.

The fact that p is also closed can be proved in the same way - the prove above works **literally** if you substitute open U with closed F (check!). It is also tempting to justify the closeness of p using Proposition 3.11 (ii) - any continuous mapping between compact and Hausdorff space is closed. But the thing is, we do not know whether $\mathbb{R}P^n$ is Hausdorff! It is actually true, but requires a proof. In this case it seems easier to prove the claim "directly" as above.

b) Mapping f is well-defined, since for every $\mathbf{x} \in \overline{B^n}$ we have that

$$1 \geq \sum_{i=1}^{n} x_i^2,$$

so the square root of $1 - \sum_{i=1}^{n} x_i^2$ is a well-defined non-negative real number. Moreover

$$x_1^2 + \ldots + x_n^2 + (1 - \sum_{i=1}^{n} x_i^2) = 1,$$

so f maps into S^n. In fact f is a **homeomorphism** of $\overline{B}^n$ to the **upper hemisphere**

$$S^n_+ = \{\mathbf{y} \in S^n \mid y_{n+1} \geq 0\}.$$

This is seen as following. By considerations above f is well-defined as a mapping of $\overline{B}^n$ into S^n_+ (last coordinate of $f(\mathbf{x})$ is non-negative, being a square root). Let $\mathrm{pr}\colon S^n_+ \to \overline{B}^n$ be (the restriction of) the projection

$$\mathrm{pr}(x_1, \dots, x_n, x_{n+1}) = \mathrm{pr}(x_1, \dots, x_n).$$

Then $\mathrm{pr} \circ f = \mathrm{id}$ directly from them definitions. Conversely, let

$$\mathbf{x} = (x_1, \dots, x_n, x_{n+1}) \in S^n_+$$

Then $\sum_{i=1}^{n+1} x_i^2 = 1$, so, since $x_{n+1} \geq 0$, we have that

$$x_{n+1} = \sqrt{1 - \sum_{i=1}^{n} x_i^2}.$$

Hence

$$f(p(x_1, \dots, x_n, x_{n+1})) = f(x_1, \dots, x_n) = (x_1, \dots, x_n, \sqrt{1 - \sum_{i=1}^{n} x_i^2}) = (x_1, \dots, x_n, x_{n+1}).$$

This means that also $f \circ \mathrm{pr} = \mathrm{id}$. We have shown that f and pr are inverses of each other. Since both are clearly continuous, we see that f is a homeomorphism, when considered as a mapping $f\colon \overline{B}^n \to S^n_+$. In particular for every $\mathbf{y} \in S^n_+$ there exists exactly one $\mathbf{x}$ such that

$$f(\mathbf{x}) = \mathbf{y}.$$

OBS The fact that f is a homeomorphism to its image is not necessary in the proof of the this exercise, but it helps and is mentioned for the convinience.

Now we go back to what we had to prove. The mapping $g = p \circ f\colon \overline{B}^n \to \mathbb{R}P^n$ is continuous since it is a composition of continuous mappings. Let us show that g is a closed surjection. Then, by Lemma 6.3., g is a quotient mapping.
Mapping $f\colon \overline{B}^n \to S^n$ is closed, since it is a continuous mapping between compact space $\overline{B}^n$ and Hausdorff space S^n. (Proposition 3.11(ii)). Mapping p is proved to be closed above in a). Hence g is closed, being a composition of two closed mappings.

Next we show that g is surjective. Notice that although p is surjective, f is not, so we cannot say that g is surjective being composition of surjective mappings, coz it's not. Instead we prove it directly. Since we will need in c) information about every inverse image $g^{-1}(a)$ for every element $a \in \mathbb{R}P^n$ (this inverse images are exactly the classes of the equiv. relation $\sim_g$), we will investigate them now. The surjectivity of g is equivalent to the claim that $g^{-1}(a)$ is not empty.

Let $a \in \mathbb{R}P^n$ be arbitrary. Then, by our definition of $\mathbb{R}P^n$

$$a = \{\mathbf{y}, -\mathbf{y}\} = \bar{\mathbf{y}}$$

for some

$$\mathbf{y} = (y_1, \ldots, y_n, y_{n+1}) \in S^n.$$

Suppose $\mathbf{x} \in \overline{B}^n$ be an element of $g^{-1}(a)$ i.e. an element for which

$$g(\mathbf{x}) = p(f(\mathbf{x})) = f(\bar{\mathbf{x}}) = \; = \bar{\mathbf{y}}.$$

Hence $\mathbf{x} \in g^{-1}(a)$ if and only if

$$f(\bar{\mathbf{x}}) = \bar{\mathbf{y}}$$

which means that either

$$f(\mathbf{x}) = \mathbf{y} \text{ or}$$
$$f(\mathbf{x}) = -\mathbf{y}.$$

Suppose first that $y_{n+1} \neq 0$. If $y_{n+1} > 0$, then $-y_{n+1} > 0$. Since $\mathbf{y}$ and $-\mathbf{y}$ both represent a in $\mathbb{R}P^n$, we may assume that $y_{n+1} > 0$. Then $\mathbf{y} \in S^n_+$ and $-\mathbf{y} \notin S^n_+$. By above we know that for $\mathbf{y}$ there exists exactly one $\mathbf{x} \in \overline{B}^n$ for which $f(\mathbf{x}) = \mathbf{y}$ and there exist not $\mathbf{x} \in \overline{B}^n$ for which $f(\mathbf{x}) = -\mathbf{y}$. The conclusion is that in case $y_{n+1} \neq 0$ the inverse image $g^{-1}a$ is a singleton, in particular not empty. This singleton consists precisely of an element $\mathbf{x} = (x_1, \ldots, x_n)$, where

$$1 - \sum_{i=1}^{n} x_i^2 = y_{n+1}^2 > 0,$$

so $\sum_{i=1}^{n} x_i^2 < 1$, which means that $\mathbf{x} \in B^n$ (open ball).

Next we investigate the case $y_{n+1} = 0$. Then both $\mathbf{y}$ and $-\mathbf{y}$ belong to S^n_+. Moreover, in this case ifg

$$f(\mathbf{x}) = \mathbf{y},$$

then

$$f(-\mathbf{x}) = -\mathbf{y}.$$

Here $\mathbf{x} \in S^{n-1}$, since

$$1 - \sum_{i=1}^{n} x_i^2 = y_{n+1}^2 = 0.$$

Since f is an injection, we conclude that in case $y_{n+1} = 0$ the inverse image $g^{-1}a$ is a two-point set $\{\mathbf{x}, -\mathbf{x}\}$, where $\mathbf{x} \in S^{n-1}$. In particular it is not empty.

We have shown that g is a closed surjection. By Lemma 6.3. g is a quotient mapping.

c) Now that we know that g is a quotient mapping, we can apply Proposition 6.5. and conclude that g induces a homeomorphism $\tilde{g} \colon \overline{B}^n / \sim_g \to \mathbb{R}P^n$. Here $\sim_g$ is the equivalence relation, whose equivalence classes are, by definition, exactly inverse images $g^{-1}a$ of all different elements $a \in \mathbb{R}P^n$. Above we have calculated what these inverse images are exactly (that's what it was for), so we see that $\sim_g$ is exactly the relation $\sim'$ given in the task, generated by the relations $(\mathbf{x}, -\mathbf{x})$, $x \in S^{n-1}$.

3. The *cone* $c(X)$ of a topological space X is a quotient space of the product space $X \times I$ with subset $X \times \{1\}$ identified to a single point (and no other identifications). In other words $c(X) = (X \times I)/X \times \{1\}$ (this notation is introduced in Example 6.9).
a) Prove that $c(S^{n-1})$ is homeomorphic to $\overline{B}^n$, for all $n \geq 1$.
b) Suppose X is compact. Prove that $c(X)$ is contractible. Is it necessary to assume that X is compact? Why/why not?

Solution: We define the mapping $g \colon S^{n-1} \times I \to \overline{B}^n$ by

$$g(\mathbf{x}, t) = (1 - t)\mathbf{x}.$$

The idea behind this mapping is that we think of $\overline{B}^n$ consisting of sphere-shaped "layers" (think onion!)

$$S_r\{|\mathbf{x}| = r\},$$

where r goes through the interval $[0,1]$. All these layers are homeomorphic to the sphere S^{n-1} in a natural way, except that S_0 is just a singleton $\{0\}$. Putting this mappings together we obtain g, which is designed so that $S^{n-1}\times\{1\}$ maps to the origin $\{0\}$.
It is easy to see that g is a surjection. Indeed

$$g(\mathbf{x},1)=\mathbf{0}$$

for any $\mathbf{x}\in S^{n-1}$ and if $\mathbf{y}\neq 0$ then

$$g(\mathbf{y}/|\mathbf{y}|,1-|\mathbf{y}|)=\mathbf{y}.$$

Mapping g is also obviously continuous. Since $S^{n-1}\times I$ is compact and $\overline{B}^n$ is Hausdorff, by Lemma 6.4. g is a quotient mapping.

Let us investigate the relation $\sim_g$. Suppose

$$g(\mathbf{x},t)=(1-t)\mathbf{x}=(1-t')\mathbf{y}=g(\mathbf{y},t'),$$

where $\mathbf{x},\mathbf{y}\in S^{n-1}$ and $0\leq t,t'\leq 1$. Taking norms we obtain

$$1-t=1-t',$$

hence $t=t'$. If $t=1$, then

$$g(\mathbf{x},1)=\mathbf{0}=g(\mathbf{y},1)$$

for all $\mathbf{x},\mathbf{y}\in S^{n-1}$. If on the other hand $t\neq 1$, then

$$(1-t)\mathbf{x}=(1-t)\mathbf{y}$$

implies that $\mathbf{x}=\mathbf{y}$. We see that the relation $\sim_g$ identifies $S^{n-1}\times\{1\}$ to a point and there are no other non-trivial identifications. Hence the quotient space $S^{n-1}\times I/\sim_g$ is exactly the cone $c(S^{n-1})$. Since g is a quotient mapping, by Proposition 6.5. g induces homeomorphism

$$\tilde{g}\colon c(S^{n-1})\to\overline{B}^n.$$

b) It is easy to come up with a natural formula for homotopy $F\colon c(X)\times I\to c(X)$ that shrinks identity mapping to a point $S^{n-1}\times\{1\}$,

$$F(\overline{(\mathbf{x},t)},s)=\overline{(\mathbf{x},t+s(1-t))}.$$

The formula $s\mapsto(1-s)t+s\cdot 1=t+s(1-t)$ is a natural linear path from t to 1 on real line.

Mapping F obviously is a homotopy from identity to the constant mapping, **once** we know it is continuous. This is surprisingly not trivial to proof, although is always true.

Consider the diagram

$$\begin{array}{ccc} (X \times I) \times I & \xrightarrow{G} & X \times I \\ \downarrow{\scriptstyle \pi \times \mathrm{id}} & & \downarrow{\scriptstyle \pi} \\ c(X) \times I & \xrightarrow{F} & c(X), \end{array}$$

where $\pi\colon X \times I \to c(X)$ is a canonical projection and $G\colon (X \times I) \times I \to X \times I$ is a homotopy

$$G((\mathbf{x}, t), s) = (\mathbf{x}, t + s(1 - t)).$$

The continuity of G is simple - it is a combination of projections and continuous mappings defined on the unit interval (and unit square). Hence from the diagram above we see that F is continuous IF we know that the mapping $\pi \times \mathrm{id}$ is a **quotient mapping** (Exercise 1). The mapping π is quotient by definition and identity mapping is always quotient, but this is not enough to conclude that $\pi \times \mathrm{id}$ is quotient - the product of two quotient mappings is not always quotient (counterexamples exist). In this case the product IS quotient, but the proof of this fact is not easy. This is a special case of the following result from general topology (you can find proof in many topology or homotopy theory texts):

Theorem. Suppose $p\colon X \to Y$ is a quotient mapping and Z a **locally compact Hausdorff space**. Then $p \times \mathrm{id}\colon X \times Z \to Y \times Z$ is a quotient mapping.

In particular theorem implies in case $Z = I$, so we can always "quotient out" homotopies to obtain homotopies between quotient spaces.

The proof of the Theorem mentioned is not exactly trivial, so we will instead prove the continuity of F in case X is a compact Hausdorff space (notice - Hausdorff assumption was unfortunately missing from the suggestion of the exercise task. In general topology it is customary to reserve term "compact" only to Hausdorff compact spaces).

We know that continuity of F follows if we prove that $\pi \times \mathrm{id}\colon (X \times I) \times I \to c(X) \times I$ is quotient. This mapping is clearly surjective and continuous. Since $(X \times I) \times I$ is compact, claim follows from Lemma 6.4. once we have shown that $c(X) \times I$ is Hausdorff. For that it is enough to prove that $c(X)$ is Hausdorff.

We conclude the proof by showing that $c(X)$ is Hausdorff whenever X is Hausdorff.
Suppose $\bar{(x,t)} \neq \bar{(y,t')}$. Suppose first $t \neq t'$. We may assume that $t < t'$. We can choose $r \in I$ such that $t < r < t'$. Then

$$U = \{(z,s) \in X \times I \mid s < r\}$$

and

$$V = \{(z,s) \in X \times I \mid s > r\}$$

are both open subsets of $X \times I$. Moreover

$$p^{-1}pU = U, p^{-1}pV = V$$

for the canonical projection $p\colon X \times I \to c(X)$. This implies that $U' = p(U)$ and $V' = p(V)$ are open in $c(X)$. It is easy to check that $U' \cap V' = \emptyset$ and by construction $\bar{(x,t)} \in U'$, $\bar{(y,t')} \in V'$. Hence U' and V' are disjoint neighbourhoods of $\bar{(x,t)}$ and $\bar{(y,t')}$.

Next suppose $t = t'$. Then we must have $t = t' < 1$, since otherwise $\bar{(x,t)} = \bar{(y,t')}$. Since we assume that X is Hausdorff there exists disjoint neighbourhoods U and V of x and y respectively in X. Let $U' = p(U \times [0,1[)$, $V' = p(V \times [0,1[)$. Then

$$p^{-1}U' = U \times [0,1[$$

is open in $X \times I$, hence U' is open in $c(X)$. Similarly one sees that V' is open in $c(X)$. Moreover U' is a nieghbourhood of $\bar{(x,t)}$, while V' is a neighbourhood of $\bar{(y,t')}$ and they do not intersect.

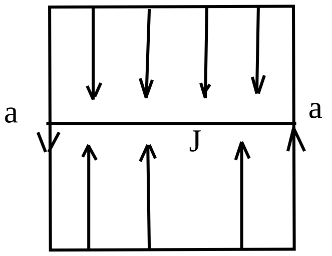
4. Prove that Mobius Band has the same homotopy type as the circle S^1.

Solution: Let us use the model $M = I^2/\sim$ for the Mobius Band, where $\sim$ is the equivalence relation in I^2 defined by relations $(x,0) \sim (1-x,1)$. Notice that this is different from the definition used in the lecture notes, where identifications are made on the vertical, not horizontal sides.

The idea of the proof is to deform the square I^2 into its "middle horizontal line"

$$J = \{(x, y) \in I^2 \mid y = 1/2\}$$

in a natural "linear way" (see the picture below). This deformation then works well with the identifications that define Mobius Band. The middle line itself in the Mobius band becomes a circle.



To formalize this idea with start with the construction of the homotopy equivalences. First consider a mapping $\alpha\colon I \to M$, where M is the Mobius band defined by

$$\alpha(t) = \overline{(t, 1/2)}.$$

Then α is continuous, since it is a composition of the continuous mapping $I \to I^2$, $t \mapsto (1/2, t)$ and the canonical projection $p\colon I^2 \to M$. Mapping α maps endpoints 0 and 1 of the unit interval to the same point, since in the Mobius band $\overline{(0, 1/2)} = \overline{(1, 1/2)}$. Hence α "quotients through" the quotient space $I/\{0, 1\}$ which is homeomorphic to S^1 (Example 6.9.2). More formally there exists a commutative diagram

$$\begin{array}{ccc} I & & \\ \downarrow{\scriptstyle \alpha} & \searrow^{q} & \\ S^1 & \xrightarrow{f} & M, \end{array}$$

where $f\colon S^1 \to M$ is the "induced mapping" defined by the formula

$$f(\cos 2\pi t, \sin 2\pi t) = (t, \bar{1}/2).$$

Here $q\colon I \to S^1$ is the quotient mapping defined by

$$q(t) = (\cos 2\pi t, \sin 2\pi t)$$

(see Example 6.9.2). Since q is quotient and α is continuous, Exercise 1 implies that f is a continuous mapping. This mapping will be a homotopy equivalence we want.

Next we construct the mapping $g\colon M \to S^1$, which will serve as a homotopy inverse of f. This mapping comes naturally from the "deformation" of the square into its vertical middle line i.e. is defined by the formula

$$g(\overline{(x,y)}) = (\cos 2\pi x, \sin 2\pi x) = q(x).$$

To justify the continuity of g we use the same reasoning as usual - mapping g is a part of the commutative diagram

$$\begin{array}{ccc} I^2 & & \\ \downarrow \beta & \searrow^{p} & \\ M & \xrightarrow{g} & S^1, \end{array}$$

where $\beta\colon I^2 \to S^1$ is defined by

$$\beta(x,y) = (\cos 2\pi x, \sin 2\pi x) = \pi(x).$$

Since β is clearly continuous and p is a quotient mapping, by Exercise 1 we obtain the continuity of g.

Next step is to show that $g \circ f$ and $f \circ g$ are actually homotopic to corresponding identity mappings. The composition $g \circ f$ is actually precisely the identity mapping of S^1. The mapping $f \circ g\colon M \to M$ is defined by the formula

$$(f \circ g)(\overline{(x,y)}) = \overline{(x, 1/2)}.$$

The homotopy between id_M and $f \circ g$ is defined by the formula

$$F(\overline{(x,y)}, t) = \overline{(x, (1-s)y + t/2)}.$$

Mapping is well defined, since

$$F(\overline{(0,y)}, t) = \overline{(0, (1-t)y + t/2)} = \overline{(1, 1-(1-t)y - t/2)} =$$

$$= \overline{(1, (1-t)(1-y) + t/2)} = F(\overline{(1, 1-y)}), y, t \in I.$$

To justify the continuity of F we consider the commutative diagram

$$\begin{array}{ccc} I^2 \times I & \xrightarrow{G} & I^2 \\ \downarrow{\scriptstyle p\times \mathrm{id}} & & \downarrow{\scriptstyle p} \\ M \times I & \xrightarrow{F} & M, \end{array}$$

where $G\colon I^2 \times I \to I^2$ is defined by

$$G((x,y),t) = (x,(1-t)y + t/2).$$

Since G is clearly continuous and p is continuous, the continuity of F follows once we know that $p \times \mathrm{id}\colon I^2 \times I \to M \to I$ is a quotient mapping. This is the same problem we have encountered in the proof of the previous exercise. In this instance we can use Lemma 6.4.. The space $I^2 \times I$ is compact and the mapping $p \times \mathrm{id}$ is clearly surjection, so it is enough to show that $M \times I$ is Hausdorff. To this end it is enough to show that the Mobius Band M is Hausdorff.

Suppose $a = \overline{(x,y)}, b = \overline{(u,v)} \in M$, $a \neq b$. Suppose first $0 < x < 1$ and $x \neq u$. Then we can find small enough $\varepsilon > 0$ such that

$$0 < x - \varepsilon < x < x + \varepsilon < 1$$

and $u \notin [x - \varepsilon, x + \varepsilon]$. Then

$$U = \{(s,t) \in I^2 \mid |x - s| < \varepsilon\},$$

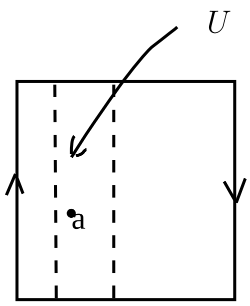
$$V = \{(s,t) \in I^2 \mid |x - s| > \varepsilon\}$$

are open subsets of I^2. Moreover U does not contain any points that have non-trivial identifications w.r.t. $\sim$ and V contains both bottom and upper horizontal sides of the square, so is closed under non-trivial identifications. It follows that

$$p^{-1}p(U) = U,$$

$$p^{-1}p(V) = V,$$

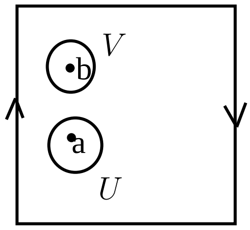
so $p(U)$ and $p(V)$ are both open in M. It is easy to see that these sets do not intersect, $p(U)$ is a neighbourhood of a and $p(V)$ is a neighbourhood of b. Hence a and b have non-intersecting neighbourhoods.



Next suppose $0 < x = u < 1$. Then $(x, y) \neq (u, v)$ and, since I^2 is Hausdorff, we can find small enough neighbourhood U of (x, y) and V of (u, v) such that
1) $U \cap V = \emptyset$,
2)

$$U \cap V \subset \{(t, s) \in I^2 \,|0 < t < 1\}.$$

The last condition assures that there the sets U and V do not contain any points with non-trivial equivalence classes, hence $p^{-1}p(U) = U$ and $p^{-1}p(V) = V$, so, as usual, $p(U)$ and $p(V)$ will be non-intersecting neighbourhoods of a and b in M.



The same arguments work if $0 < u < 1$. The last case is when $x, u \in \{0, 1\}$. Interchanging y or v with $1 - y$ or $v - 1$, if necessary, we may assume that $x = u = 0$. Then $y \neq v$.

We choose $\varepsilon > 0$ such that open ε-ball neighbourhoods **in** M

$$B((0, y), \varepsilon) = \{(s, t) \in I^2 \mid |(0, y) - (s, t)| < \varepsilon\},$$

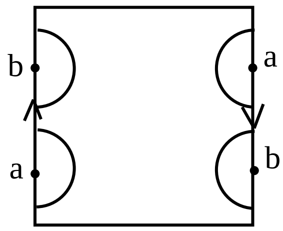
$$B((0, v), \varepsilon) = \{(s, t) \in I^2 \mid |(0, v) - (s, t)| < \varepsilon\}$$

do not intersect and $\varepsilon < 1/2$. Then corresponding neighbourhoods "on the other side"

$$B((1, 1 - y), \varepsilon) = \{(s, t) \in I^2 \mid |(1, 1 - y) - (s, t)| < \varepsilon\},$$

$$B((1, 1 - v), \varepsilon) = \{(s, t) \in I^2 \mid |(1, 1 - v) - (s, t)| < \varepsilon\}$$

also do not intersect.



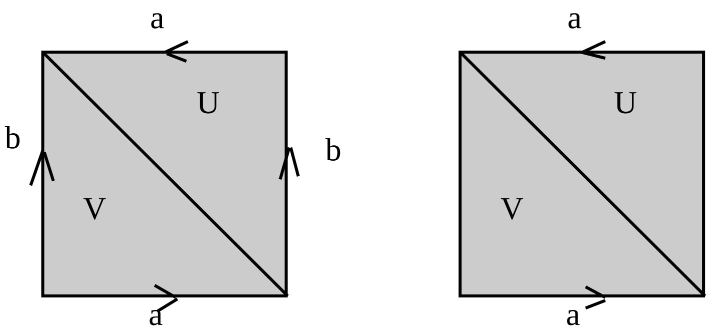
Hence if we define

$$U = B((0,u),\varepsilon) \cup B((1,1-u),\varepsilon),$$

$$V = B((0,v),\varepsilon) \cup B((1,1-v),\varepsilon),$$

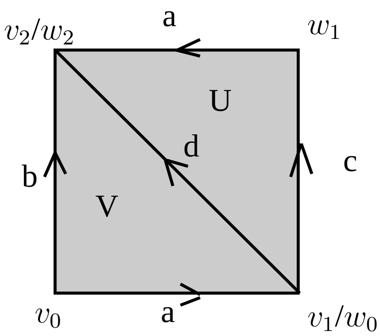
then, as usual $p^{-1}p(U) = U, p^{-1}p(V) = V$, so $p(U)$ and $p(V)$ are disjoint neighbourhoods of a and b in this case. The proof is finished.

5. Define an exact representation of
   a) the Mobius Band
   b) the Klein's bottle
   as a polyhedron of a Δ-complex with two triangles, based on the picture below. Remember to order the simplices and define identifications!



Klein's bottle Mobius's Band

Solution: Here is one way to triangulate Mobius Band as the polyhedron of Δ-complex:



The complex has two 2-simplices $U = [\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2]$ and $V = [\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2]$. Sides $[\mathbf{v}_0, \mathbf{v}_1]$ and $[\mathbf{w}_1, \mathbf{w}_2]$, indicated by a, are identified. Also the sides $[\mathbf{v}_1, \mathbf{v}_2]$ and $[\mathbf{w}_0, \mathbf{w}_2]$ which give common side d representing diagonal are identified. As a consequence we also have the following identifications on the vertices:

$$\mathbf{v}_0 \sim \mathbf{w}_1$$

$$\mathbf{v}_1 \sim \mathbf{w}_2$$

$$\mathbf{v}_1 \sim \mathbf{w}_0$$

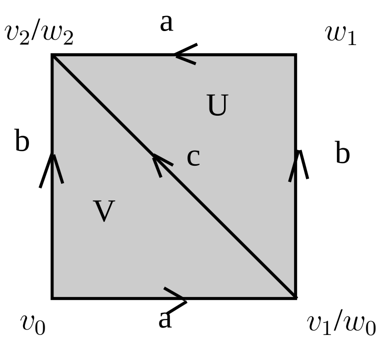
$$\mathbf{v}_2 \sim \mathbf{w}_2$$

This also forces identifications

$$\mathbf{v}_1 \sim \mathbf{v}_2$$

$$\mathbf{w}_0 \sim \mathbf{w}_2$$

b) A way to triangulate Klein's bottle:



This complex has two triangles $U = [\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2]$ and $V = [\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2]$. Sides $[\mathbf{v}_0, \mathbf{v}_1]$ and $[\mathbf{w}_1, \mathbf{w}_2]$, indicated by a, are identified. Sides $[\mathbf{v}_0, \mathbf{v}_2]$ and $[\mathbf{w}_0, \mathbf{w}_1]$, indicated by b, are identified. Also the sides $[\mathbf{v}_1, \mathbf{v}_2]$ and $[\mathbf{w}_0, \mathbf{w}_2]$, which give common side d representing diagonal are identified. As a consequence we also have the following identifications on the vertices:

$$\mathbf{v}_0 \sim \mathbf{w}_1$$
$$\mathbf{v}_1 \sim \mathbf{w}_2$$
$$\mathbf{v}_0 \sim \mathbf{w}_0$$
$$\mathbf{v}_2 \sim \mathbf{w}_1$$
$$\mathbf{v}_1 \sim \mathbf{w}_0$$
$$\mathbf{v}_2 \sim \mathbf{w}_2$$

This also forces identifications

$$\mathbf{v}_0 \sim \mathbf{v}_1 \sim \mathbf{v}_2$$

$$\mathbf{w}_0 \sim \mathbf{w}_1 \sim \mathbf{w}_2$$

In other words this complex has only one vertex - all vertices are identified.

6. Consider the quotient space $X = S^1 \times I/\sim$, where $\sim$ is generated by all relations of the form $(x, 0) \sim (-x, 0)$, $x \in S^1$.
a) Present X as a polyhedron of a Δ-complex (Advice: start with the square). Drawing with arrows suffices.
b) Show that X is actually homeomorphic to the Mobius band (use "cut and glue" technique, see Exercise 6.18).

Solution: a) The space $S^1 \times I$ can be thought of as the space obtained from the square I^2 by identifying $(0, x) \sim (1, x)$ for all $x \in I$. The quotient mapping $I^2 \to S^1 \times I$ is defined by

$$(x, y) \mapsto (e^{2\pi x i}, y),$$

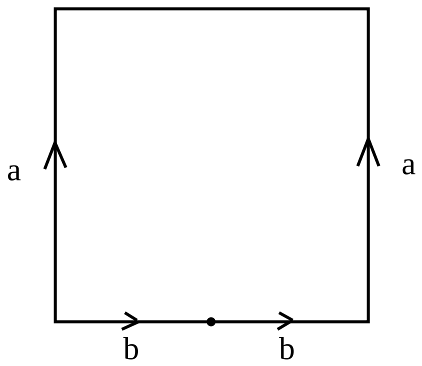
where we use complex exponential notation -

$$e^{2\pi x i} = (\cos 2\pi x, \sin 2\pi x).$$

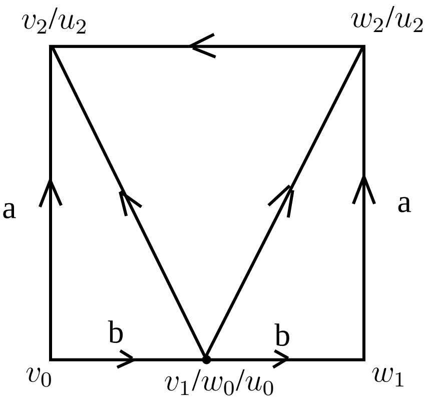
Under this identification the bottom $S^1 \times \{0\}$ becomes in the square the part on the x-axis i.e. the set $I \times \{0\}$. The identifications $(x, 0) \sim (-x, 0)$, $x \in S^1$ become

$$(x, 0) \sim (x + 1/2, 0), 0 \leq x \leq 1/2.$$

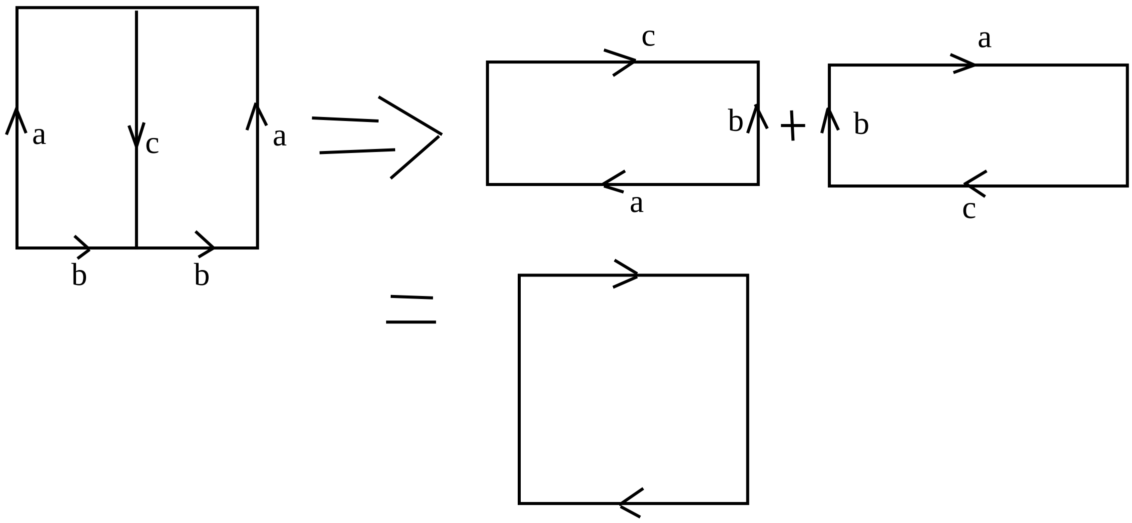
Hence we can draw X schematically using the square as following:



Adding extra lines in order to obtain triangles, we obtain scheme for Δ-complex representing X, as following:



b) First we divide the square with our identification along the vertical line $x = 1/2$. Then we rearrange pieces.



Mobius band!

7.* (bonus exercise)
Use exercise 3.3 to define a triangulation of the projective plane $\mathbb{R}P^n$ as a polyhedron of a Δ-complex, for all $n \geq 1$. The triangulation should have 2^n geometrical simplices in dimension n.

Solution: The triangulation of the sphere S^n from Exercise 3.3. consisted of simplices of all simplices of the form

$$[\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_n, \pm\mathbf{e}_{n+1}]$$

and all their faces. The relation $\sim$ defined on these simplices by

$$[\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}_{n+1}] \sim [-\mathbf{v}_1, \ldots, -\mathbf{v}_n, -\mathbf{v}_{n+1}]$$

defines a Δ-complex (forced relations on faces similar), whose polyhedron is easily seen to be $\mathbb{R}P^n$.