## Department of Mathematics and Statistics
## Introduction to Algebraic topology, fall 2013
### Exercises 3 - Solutions

1. a) Prove that the standard simplex

$$\Delta_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i, \sum_{i=1}^{n} x_i \leq 1\}$$

is a closed and bounded, hence compact, subset of $\mathbb{R}^n$.
b) Show that the topological interior of the standard simplex $\Delta_n$ with respect to $\mathbb{R}^n$ coincides with its simplicial interior $\operatorname{Int}\sigma$, and the same is true for topological/simplicial boundaries.

**Solution:** a) First we show that $\Delta_n$ is closed in $\mathbb{R}^n$. Consider the mappings $pr_j \colon \mathbb{R}^n \to \mathbb{R}$, $g \colon \mathbb{R}^n \to \mathbb{R}$, $j = 1, \dots, n$, defined by

$$pr_j(x_1, \dots, x_j, \dots, x_n) = x_j,$$

$$g(x_1, \dots, x_j, \dots, x_n) = \sum_{i=1}^{n} x_i.$$

From the basic topology and/or calculus courses we know that these mappings are continuous. Indeed mappings $pr_j$ are just (linear) projections and $g$ is a sum of these projections (the sum of continuous real-valued functions is continuous).
Inverse images of closed sets with respect to continuous mappings are closed (Lemma 3.2.), so the subsets

$$F_j = pr_j^{-1}([0, \infty[) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_j \geq 0\}, j = 1, \dots, n,$$

$$F_{j+1} = g^{-1}(]-\infty, 1]) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i \leq 1\}$$

of $\mathbb{R}^n$ are all closed. Since

$$\Delta_n = \bigcap_{i=1}^{n+1} F_j,$$

simplex $\Delta_n$ is closed as an intersection of closed sets.

Next we present three ways to see that $\Delta_n$ is bounded.

**Proof 1**: Direct straightforward estimate. Let $\mathbf{x} = (x_1, \ldots, x_n) \in \Delta_n$. Then$x_i \geq 0$ for all $i = 1, \ldots, n$ and $\sum_{i=1}^n x_i \leq 1$. This implies that

$$0 \leq x_i \leq \sum_{i=1}^n x_i \leq 1$$

for all $i = 1, \ldots, n$. Hence

$$|\mathbf{x}|^2 = \sum_{i=1}^n x_i^2 \leq \sum_{i=1}^n 1^2 = n,$$

in other words $|\mathbf{x}| \leq \sqrt{n}$. This is true for every $\mathbf{x} \in \Delta_n$.

**Proof 2**: A better estimate follows from observation that since we already know that $0 \leq x_i \leq 1$ for all $i = 1, \ldots, n$, when $\mathbf{x} \in \Delta_n$, then in particular

$$x_i^2 \leq x_i \leq 1$$

for all $i = 1, \ldots, n$. Hence

$$|\mathbf{x}|^2 = \sum_{i=1}^n x_i^2 \leq \sum_{i=1}^n x_i = 1,$$

in other words $|\mathbf{x}| \leq 1$. This is true for every $\mathbf{x} \in \Delta_n$.

**Proof 3**: Finally there is an abstract way to obtain the inclusion

$$\Delta_n \subset \overline{B}^n(\overline{0}, 1)$$

directly, using the theory of convex sets. Indeed all the vertices $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n$ of a simplex $\Delta_n$ belong to the closed unit ball $\overline{B}^n(\overline{0}, 1)$ centred at origin. This ball is convex. Since $\Delta_n$ by definition is **the smallest** convex set containing its vertices, we obtain the inclusion

$$\Delta_n \subset \overline{B}^n(\overline{0}, 1).$$

b) It is easy to verify that the simplicial interior of $\Delta_n$ is exactly the set

$$\text{Int}\,\Delta_n = U = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i > 0 \text{ for all } i, \sum_{i=1}^n x_i < 1\}.$$

Using the mappings $pr_j\colon \mathbb{R}^n \to \mathbb{R}$, $g\colon \mathbb{R}^n \to \mathbb{R}$, $j = 1, \ldots, n$,

$$pr_j(x_1, \ldots, x_j, \ldots, x_n) = x_j,$$

$$g(x_1, \ldots, x_j, \ldots, x_n) = \sum_{i=1}^{n} x_i$$

already defined in a) above, we see that we can represent $U$ as a finite intersection

$$U = \bigcap_{i=1}^{n+1} V_j,$$

where

$$V_j = pr_j^{-1}(]0, \infty[) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_j > 0\}, j = 1, \ldots, n,$$

$$V_{j+1} = g^{-1}(] - \infty, 1]) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i < 1\}$$

are open as the inverse images of open subsets of $\mathbb{R}$ under continuous mappings. Since a finite intersection of open sets is open, $U$ is open in $\mathbb{R}^n$. Hence $U$ is an open subset of $\Delta_n$. Topological interior is **the biggest** open subset of $\Delta_n$ (Proposition 3.17(1)), so this implies that

$$U = \operatorname{Int} \Delta_n \subset \operatorname{int} \Delta_n.$$

Next we prove the converse inclusion $\operatorname{int} \Delta_n \subset \operatorname{Int} \Delta_n$. Suppose $\mathbf{x} = (x_1, \ldots, x_n) \in \operatorname{int} \Delta_n$, where the interior is with respect to $\mathbb{R}^n$. We have to show that $x_i > 0$ for all $i = 1, \ldots, n$ and that $\sum_{i=1}^{n} x_i < 1$. We do this using counter-assumptions. Suppose $x_i = 0$ for some $i = 1, \ldots, n$. Then, for every positive $\varepsilon$, an $\varepsilon$-neighbourhood of $\mathbf{x}$ obviously contains a point

$$\mathbf{x} - \varepsilon_i/2 = (x_1, \ldots, -\varepsilon/2, \ldots, x_n),$$

which is not an element of $\Delta_n$ (one of the coordinates is negative). This contradicts the assumption $\mathbf{x} \in \operatorname{int} \Delta_n$. Hence we must have $x_i > 0$ for all $i = 1, \ldots, n$.

Suppose $\sum_{i=1}^{n} x_i = 1$ (counter-assumption). Then for every positive $\varepsilon$, an $\varepsilon$-neighbourhood of $\mathbf{x}$ obviously contains a point

$$\mathbf{x} + \varepsilon_1/2 = (x_1 + \varepsilon/2, \ldots, x_n),$$

which is not an element of $\Delta_n$, since for this point the sum of coordinates is

$$\sum_{i=1}^{n} x_i + \varepsilon/2 = 1 + \varepsilon/2 > 1.$$

Again, we obtain the contradiction with the assumption $\mathbf{x} \in \operatorname{int} \Delta_n$. Thus we also must have $\sum_{i=1}^{n} x_i < 1$. We have shown that every point of $\operatorname{int} \Delta_n$ belongs to the simplicial interior $U$ of $\Delta_n$.

2. Suppose $C$ is a compact convex subset of $\mathbb{R}^n$ such that $\mathbf{0} \in \operatorname{int} C$. Let $f\colon \partial C \to S^{n-1}$ be the mapping

$$f(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|},$$

which we have shown to be a homeomorphism in the proof of Theorem 3.20. Prove that the mapping $G\colon \overline{B}^k \to C$ defined by

$$G(\mathbf{t}) = \begin{cases} |\mathbf{t}| \cdot \left(f^{-1}\frac{\mathbf{t}}{|\mathbf{t}|}\right) \text{ if } \mathbf{t} \neq \mathbf{0}, \\ \mathbf{0}, \text{ if } \mathbf{t} = \mathbf{0} \end{cases}$$

is a continuous bijection.

**Solution:** Let us start by showing that $G$ is actually well-defined, i.e. $G(\mathbf{t}) \in C$ for all $\mathbf{t} \in \overline{B}^k$. If $\mathbf{t} = \mathbf{0}$, then $G(\mathbf{t}) = \mathbf{0} \in C$ by assumption. Suppose $\mathbf{t} \neq \mathbf{0}$. Then, by assumptions on the mapping $f$, the element $f^{-1}(\frac{\mathbf{t}}{|\mathbf{t}|}) = \mathbf{y}$ is well-defined (since $\frac{\mathbf{t}}{|\mathbf{t}|} \in S^{k-1}$ and is an element of $\partial C \subset C$ (last inclusion - because $C$ is closed). Also, $t = |\mathbf{t}| \in ]0, 1]$, so

$$G(\mathbf{t}) = t \cdot \mathbf{y} = (1-t)\mathbf{0} + t \cdot \mathbf{y} \in C$$

by the convexity of $C$. We have shown that $G$ is well-defined.

Next we show that $G$ is continuous. It is clear that the restriction of $G$ on the open subset $\overline{B}^k \setminus \{\mathbf{0}\}$ (the punctured ball) is continuous (its formula is a combination of continuous operations including $f^{-1}$. It follows that $G$ is continuous at every point of $\overline{B}^k \setminus \{\mathbf{0}\}$ (openess of this set is essential here, a mapping the restriction of which is continuous in a **neighbourhood** of a point is continuous at this point). It remains to show the continuity of $G$ in the origin. Let $\mathbf{t} \in \overline{B}^k$. Then

$$|G(\mathbf{t}) - G(\mathbf{0})| = |G(\mathbf{t})| = |\mathbf{t}||f^{-1}\frac{\mathbf{t}}{|\mathbf{t}|}| \leq K|\mathbf{t}| < \varepsilon,$$

when $|\mathbf{t}| < \varepsilon/K$. Here $K > 0$ is chosen so that

$$C \subset B(\mathbf{0}, K).$$

Such $K$ exists because $C$ is assumed to be bounded. This calculation implies that $G$ is also continuous at origin, so we are done with continuity.

Next we show injectivity of $G$. It is clear that only origin maps to origin. Indeed if $\mathbf{t} \neq \mathbf{0}$, then both $|\mathbf{t}| \neq 0$ and $f^{-1}\frac{\mathbf{t}}{|\mathbf{t}|} \neq \mathbf{0}$, being an element of the boundary $\partial C$ (which do not contain origin, since we are assuming that origin is an interior point of $C$).

Suppose

$$G(\mathbf{t}) = |\mathbf{t}| \cdot \left(f^{-1}\frac{\mathbf{t}}{|\mathbf{t}|}\right) = \mathbf{x} = |\mathbf{s}| \cdot \left(f^{-1}\frac{\mathbf{s}}{|\mathbf{s}|}\right) = G(\mathbf{s})$$

for some $\mathbf{t}, \mathbf{s} \in \overline{B}^k \setminus \{\mathbf{0}\}$. Then

$$\frac{\mathbf{x}}{|\mathbf{t}|} = f^{-1}\frac{\mathbf{t}}{|\mathbf{t}|} \text{ and}$$

$$\frac{\mathbf{x}}{|\mathbf{s}|} = f^{-1}\frac{\mathbf{s}}{|\mathbf{s}|}.$$

By definition $f^{-1}$ maps onto $\partial C$, so both $\mathbf{x}/|\mathbf{t}|$ and $\mathbf{x}/|\mathbf{s}|$ belong to the boundary of $C$. According to Lemma 3.19 (applied to $\mathbf{0} \in \operatorname{int} C$), however, there exist **unique** $a > 0$ such that a point of the form $a\mathbf{x} \in \partial C$. This implies that

$$|\mathbf{t}| = |\mathbf{s}|,$$

which in turn implies that

$$f^{-1}\frac{\mathbf{t}}{|\mathbf{t}|} = \frac{\mathbf{x}}{|\mathbf{t}|} = \frac{\mathbf{x}}{|\mathbf{s}|} = f^{-1}\frac{\mathbf{s}}{|\mathbf{s}|}.$$

Being an inverse of a bijection, $f^{-1}$ is a bijection itself, in particular injection. Hence we have that

$$\frac{\mathbf{t}}{|\mathbf{t}|} = \frac{\mathbf{s}}{|\mathbf{s}|} = \frac{\mathbf{s}}{|\mathbf{t}|}$$

so $\mathbf{t} = \mathbf{s}$. We have shown that $G$ is an injection.

Next we show that $G$ is a surjection. Let $\mathbf{x} \in C$ be arbitrary. If $\mathbf{x} = \mathbf{0}$, then $G(\mathbf{0}) = \mathbf{x}$. Suppose $\mathbf{x} \neq \mathbf{0}$. By Lemma 3.19 there exist unique $r \in ]0, 1]$ and $\mathbf{y} \in \partial C$ such that $\mathbf{x} = r\mathbf{y}$. Then

$$\mathbf{t} = r\frac{\mathbf{x}}{|\mathbf{x}|}.$$

is an element of the closed ball $\overline{B}^k$ and

$$\frac{\mathbf{t}}{|\mathbf{t}|} = \frac{\mathbf{x}}{|\mathbf{x}|} = \frac{\mathbf{y}}{|\mathbf{y}|},$$

$$|\mathbf{t}| = r.$$

It follows that

$$f(\mathbf{y}) = \frac{\mathbf{y}}{|\mathbf{y}|} = \frac{\mathbf{t}}{|\mathbf{t}|},$$

so

$$f^{-1}(\frac{\mathbf{t}}{|\mathbf{t}|}) = \mathbf{y}.$$

This implies that

$$G(\mathbf{t}) = |\mathbf{t}|f^{-1}(\frac{\mathbf{t}}{|\mathbf{t}|}) = r\mathbf{y} = \mathbf{x}.$$

The surjectivity is proved.

3. Let $K_0$ be the set consisting of all possible sets of the form

$$\text{conv}(\mathbf{e}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n) \subset \mathbb{R}^n,$$

where $\mathbf{v}_i \in \{\mathbf{e}_i, -\mathbf{e}_i\}$ for $i = 1, \ldots, n$.
a) Show that $K_0$ is a collection of simplices of $\mathbb{R}^n$, but is not a simplicial complex.
b) Let $K$ be the collection of all faces of simplices in $K_0$. Show that $K$ is a simplicial complex. What is a polyhedron $|K|$ of $K$?
c) Show that $K$ has a subcomplex $L$ such that the (topological) boundary of $|K|$ in $\mathbb{R}^n$ is a polyhedron $|L|$ of $L$. How many $(n-1)$-dimensional simplices $L$ contains?

**Solution:** a) First we need to show that every sequence of the form $(\mathbf{e}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n)$, where $\mathbf{v}_i \in \{\mathbf{e}_i, -\mathbf{e}_i\}$ is affinely independent. By Lemma 2.10 this is equivalent to the linear independence of the sequence

$$(\mathbf{v}_1 - \mathbf{e}_0, \ldots, \mathbf{v}_n - \mathbf{e}_0).$$

This is simply the sequence $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and the fact that it is linearly independent is simple linear algebra exercise.

Hence $K_0$ is indeed a collection of simplices. It is obviously not closed under faces (unless we are talking about the degenerate case $n = 0$), so it cannot be a simplicial complex.

b) $K$ is obviously closed under faces by its definition (a face of a face of $\sigma$ is a face of $\sigma$ itself). To prove that $K$ is a simplicial complex it is enough, by Lemma 4.2., to show that every point $\mathbf{x}$ of the union

$$|K| = \bigcup_{\sigma \in K} \sigma$$

belongs to the simplicial interior $\operatorname{Int}\sigma$ for **unique** $\sigma \in K$. Since we are asked to determine the polyhedron $|K|$ anyway, we start by examining what is $|K|$. Suppose $\mathbf{x} \in |K|$. Then $\mathbf{x} \in \sigma$ for some $\sigma \in K$ and since every such a simplex is a face of some simplex in $K_0$, we might as well assume that $\sigma \in K_0$. Then

$$\mathbf{x} = t_0\mathbf{e}_0 + t_1\mathbf{v}_1 + \ldots + t_n\mathbf{v}_n$$

for some (unique) scalars $t_0, \ldots, t_n \geq 0$, $t_0 + \ldots + t_n = 1$. Since $\mathbf{v}_i \in \{\mathbf{e}_i, -\mathbf{e}_i\}$ for all $i = 1, \ldots, n$, this equation implies that

$$(x_1, \ldots, x_n) = \mathbf{x} = (\pm t_1, \pm t_2, \ldots, \pm t_n).$$

Since all scalars $t_i$ are non-negative, this is equivalent to $t_i = |x_i|$ for all $i = 1, \ldots, n$. It follows that

$$\sum_{i=1}^{n} |x_i| = \sum_{i=1}^{n} t_i = 1 - t_0 \leq 1.$$

In other words we have shown that

$$|K| \subset \{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i=1}^{n} |x_i| \leq 1\} = X.$$

We show the opposite inclusion by showing that every point $\mathbf{x} \in X$ belongs to the simplicial interior of **exactly one** simplex of $K$. This will also automatically then conclude the proof of the claim that $K$ is a simplicial complex.

Thus, let $\mathbf{x} = (x_1, \ldots, x_n)$ be an element of $X$, which means precisely that

$$\sum_{i=1}^{n} |x_i| \leq 1.$$

Now if $\mathbf{x}$ belongs to the simplex $\sigma$ with vertices $(\mathbf{e}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n)$, where $\mathbf{v}_i \in \{\mathbf{e}_i, -\mathbf{e}_i\}$ for $i = 1, \ldots, n$, then

$$\mathbf{x} = t_0\mathbf{e}_0 + t_1\mathbf{v}_1 + \ldots + t_n\mathbf{v}_n$$

for some (unique) scalars $t_0, \ldots, t_n \geq 0$, $t_0 + \ldots + t_n = 1$, which, as we have seen above implies that $t_i = |x_i|$ for all $i = 1, \ldots, n$. Moreover if $t_i > 0$, $i = 1, \ldots, n$ we must then have that $\mathbf{v}_i = \mathbf{e}_i$ exactly when $x_i > 0$ and $\mathbf{v}_i = -\mathbf{e}_i$ exactly when $x_i < 0$. Finally

$$t_0 = 1 - \sum_{i=1}^{n} t_i = 1 - \sum_{i=1}^{n} |x_i|,$$

so $t_0 = 0$ if and only if

$$\mathbf{x} \in \partial X = \{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i=1}^{n} |x_i| = 1\}$$

(the fact that the topological boundary of $X$ with respect to $\mathbb{R}^n$ is exactly this set is proved similarly to the proofs concerning interior and boundary of a standard simplex in exercise 1b above).

It follows that
1) $\mathbf{x}$ belongs to the interior of face of $\sigma$ which vertices include $\mathbf{e}_i$ if and only if $x_i > 0$, $-\mathbf{e}_i$ if and only if $x_i > 0$, $i = 1, \ldots, n$, and also by $\mathbf{e}_0$ if and only if $\mathbf{x} \in \partial X$.

2) Suppose $\mathbf{x}$ belongs to the interior of a face of a simplex $\sigma$ of $K_0$. Then this simplex is exactly the simplex spanned by the vectors $\mathbf{e}_i$ if and only if $x_i > 0$, $-\mathbf{e}_i$ f if and only if $x_i > 0$, $i = 1, \ldots, n$, and also by $\mathbf{e}_0$ if and only if $\mathbf{x} \in \partial X$.

We have both shown that

$$|K| = \{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i=1}^{n} |x_i| \leq 1\}.$$

and proved that $K$ is a simplicial complex.

c) The topological boundary $\partial|K|$ of the polyhedron $|K|$ is a subset

$$\{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i=1}^{n} |x_i| = 1\}.$$

By the considerations above this set is exactly a polyhedron $|L|$ of the simplicial subcomplex $L$ of $K$ spanned by all the faces of simplices that have the form $\operatorname{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$, where $\mathbf{v}_i \in \{\mathbf{e}_i, -\mathbf{e}_i\}$ for $i = 1, \ldots, n$. In other words $L$ is defined as $K$, but without references to the origin $\mathbf{e}_0$.

Since for every $i = 1, \ldots, n$ there is exactly 2 choices for $\mathbf{v}_i$, $L$ contains exactly $2^n$ simplices of dimension $(n-1)$.

**Note:** $K$ is a triangulation of the closed ball $\overline{B}^n$ and $L$ is a triangulation of the sphere $S^{n-1}$.

4. Suppose $K$ is a simplicial complex and let $\mathbf{x} \in |K|$. By Lemma 4.2. there exists a unique simplex $\operatorname{car}(\mathbf{x}) \in K$ which contains $\mathbf{x}$ as an interior point.
We also define **the star** of $\mathbf{x}$ to be the union of all simplicial interiors of simplices that contain $\mathbf{x}$, in other words

$$\operatorname{St}(\mathbf{x}) = \bigcup \{\operatorname{Int} \sigma \mid \mathbf{x} \in \sigma\}.$$

Denote the vertices of $\operatorname{car}(x)$ by $\mathbf{v}_0, \ldots, \mathbf{v}_n$. Prove that

a) $\operatorname{St}(\mathbf{x})$ is an open neighbourhood of $\mathbf{x}$ in $|K|$.

b)

$$\operatorname{St}(\mathbf{x}) = \bigcup \{\operatorname{Int} \sigma \mid \operatorname{car}(\mathbf{x}) < \sigma\} = \bigcup \{\operatorname{Int} \sigma \mid \mathbf{v}_0, \ldots, \mathbf{v}_n \text{ are vertices of } \sigma\}.$$

c)

$$\operatorname{St}(\mathbf{x}) = \bigcap_{i=0}^{n} \operatorname{St}(\mathbf{v}_i).$$

**Solution:** a) Let us define the subset of $K$

$$L = \{\sigma \in K \mid \mathbf{x} \notin \sigma\}.$$

The subset $L$ is obviously closed under faces (if a simplex does not contain $\mathbf{x}$, any face of it cannot contain $\mathbf{x}$ either). Thus $L$ is a **subcomplex** of $K$. We show that

$$\operatorname{St}(\mathbf{x}) = |K| \setminus |L|.$$

This would imply that $\operatorname{St}(\mathbf{x})$ is open in $|K|$, since, by Lemma 4.7, a polyhedron $|L|$ of a subcomplex $L$ of $K$ is always closed in $|K|$.

Suppose $\mathbf{y} \in |K|$ is arbitrary. Then, by Lemma 4.2. there exists **unique** simplex $\sigma$ of $K$ such that $\mathbf{y} \in \operatorname{Int}\sigma$. There are exactly two mutually exclusive possibilities -
1) either $\mathbf{x} \in \sigma$ or
2) $\mathbf{x} \notin \sigma$.
We will show that $\mathbf{y} \in \operatorname{St}(\mathbf{x})$ if and only if case 1) is true and $\mathbf{y} \in |L|$ if and only if the case 2) is true. This will prove that $|K|$ is a disjoint union of the sets $\operatorname{St}(\mathbf{x})$ and $|L|$, so $\operatorname{St}(\mathbf{x}) = |K| \setminus |L|$.

Since, by Lemma 4.7, the simplicial interiors of the simplices of a simplicial complex are always disjoint, the inclusion $\mathbf{y} \in \operatorname{St}(\mathbf{x})$ is true if and only if $\mathbf{x} \in \sigma$ where $\sigma$ is the unique simplex that contains $\mathbf{y}$ as an interior point. Hence $\mathbf{y} \in \operatorname{St}(\mathbf{x})$ if and only if $\mathbf{x} \in \sigma$.

To prove the second claim we first assume that $\mathbf{x} \notin \sigma$. Then, by the definition of $L$, the simplex $\sigma \in L$, and, since $\mathbf{y} \in \sigma$, it follows that $\mathbf{y} \in |L|$.

Conversely suppose $\mathbf{y} \in |L|$. Then there exists a simplex $\sigma'$ such that $\mathbf{y} \in \sigma'$ and $\mathbf{x} \notin \sigma'$ (i.e. $\sigma' \in L$). Now consider an intersection $\sigma \cap \sigma'$. It is not empty, since $\mathbf{y}$ belongs to it. By the definition of the simplicial complex this intersection thus has to be a common face of both $\sigma$ and $\sigma'$. On the other hand this intersection contains $\mathbf{y}$, which is an interior point of $\sigma$. The only face of $\sigma$ which intersects the interior of $\sigma$ is $\sigma$ itself. Hence $\sigma \cap \sigma' = \sigma$, in particular $\sigma \subset \sigma'$. It follows that $\mathbf{x} \notin \sigma$, since otherwise $\mathbf{x} \in \sigma'$, which contradicts our assumptions. Hence we have shown that $\mathbf{x} \notin \sigma$ if and only if $\mathbf{y} \in |L|$, which is what we wanted to prove. The claim

$$\operatorname{St}(\mathbf{x}) = |K| \setminus |L|$$

is now proved, and, as we already noticed, this implies that $\operatorname{St}(\mathbf{x})$ is open.

Clearly $\mathbf{x}$ belongs to $\operatorname{St}(\mathbf{x})$, since there exists (unique) simplex $\sigma$ of $K$ such that $\mathbf{x} \in \operatorname{Int}\sigma$ and then

$$\mathbf{x} \in \operatorname{Int}\sigma \subset \operatorname{St}(\mathbf{x}).$$

b) The equation

$$\operatorname{St}(\mathbf{x}) = \bigcup \{\operatorname{Int}\sigma \mid \operatorname{car}(\mathbf{x}) < \sigma\}$$

obviously follows once we show that an arbitrary simplex $\sigma$ of $K$ contains $\mathbf{x}$ if and only if $\sigma' = \operatorname{car}(\mathbf{x})$ is a face of $\sigma$. This is the same argument we have already seen in the proof of a) above. Namely if $\sigma' \subset \sigma$, then trivially

$$\mathbf{x} \in \sigma' \subset \sigma,$$

so $\mathbf{x} \in \sigma$. Conversely suppose $\mathbf{x} \in \sigma$. Then the intersection $\sigma' \cap \sigma$ is non-empty, so is a common face of both $\sigma$ and $\sigma'$. On the other hand this intersection contains $\mathbf{x}$, which is an interior point of $\sigma'$. The only face of $\sigma'$ which intersects the interior of $\sigma'$ is $\sigma$ itself. Hence $\sigma \cap \sigma' = \sigma'$, in particular $\sigma'$ is a face of $\sigma$ which is what we had to show. The equation

$$\bigcup\{\operatorname{Int}\sigma \mid \operatorname{car}(\mathbf{x}) < \sigma\} = \bigcup\{\operatorname{Int}\sigma \mid \mathbf{v}_0, \ldots, \mathbf{v}_n \text{ are vertices of } \sigma\}$$

is obvious, since $\mathbf{v}_0, \ldots, \mathbf{v}_n$ are exactly the vertices of $\operatorname{car}(\mathbf{x})$, by assumption.

c) Suppose $\sigma$ is a simplex of $K$. As we have already seen above $\sigma$ contains $\mathbf{x}$ if and only if $\operatorname{car}(\mathbf{x})$ is a face of $\sigma$ i.e. if and only if $\mathbf{v}_0, \ldots, \mathbf{v}_n$ are all vertices of $\sigma$. Vertex $\mathbf{v}$ belongs to $\sigma$ if and only if

$$\operatorname{Int}\sigma \subset \operatorname{St}(\mathbf{v}).$$

It follows that

$$\operatorname{Int}\sigma \subset \bigcap_{i=0}^{n} \operatorname{St}(\mathbf{v}_i)$$

if and only if $\mathbf{v}_0, \ldots, \mathbf{v}_n$ are all vertices of $\sigma$. Since the simplicial interiors are disjoint, this implies that

$$\operatorname{St}(\mathbf{x}) = \bigcap_{i=0}^{n} \operatorname{St}(\mathbf{v}_i).$$

5. The covering $\mathbf{X} = (X_i)_{i \in I}$ of the topological space $X$ is called *locally finite* if every point $x \in X$ has a neighbourhood $U$, which intersects only a finite amount of the elements of the covering $\mathbf{X}$. Formally this means that the subset $J$ of the index set $I$ defined by

$$J = \{i \in I \mid U \cap X_i \neq \emptyset\}$$

is finite. Covering $\mathbf{X}$ is called *closed* if all elements of $\mathbf{X}$ are closed in $X$.

Prove that is $\mathbf{X} = (X_i)_{i\in I}$ is a closed and locally finite covering of $X$, then the topology of $X$ is coherent with the family $\mathbf{X}$.

Give an example of a closed covering $\mathbf{X}$ of a topological space $X$ such that the topology of $X$ is not coherent with $\mathbf{X}$.

**Solution:** Suppose $V$ is a subset of $X$ such that $V \cap X_i$ is open in $X_i$ for all $i \in I$. We need to show that $V$ is open in $X$. It is enough to show that $F = X \setminus V$ is closed in $X$. This is equivalent to showing that $\overline{F} \subset F$.

Suppose $x \in \overline{F}$. Let $U$ be a neighbourhood of $x$ that intersects only a finite amount of the sets $X_i$. Suppose $W$ is an arbitrary neighbourhood of $x$. Then $U \cap W$ is also a neigbourhood of $x$, so it intersects $F$. It follows that an arbitrary neighbourhood $W$ of $x$ intersects $F \cap U$. In other words

$$x \in \overline{F \cap U}.$$

Since $V \cap X_i$ is open in $X_i$, its compliment in $X_i$

$$F_i = X_i \setminus (U \cap X_i) = (X \setminus U) \cap X_i = F \cap X_i$$

is closed in $X_i$ for every $i \in I$. This means that

$$F_i = X_i \cap G_i,$$

where $G_i$ is some closed subset of $X$. However $X_i$ is assumed to be closed for every $i \in I$, so the intersection $X_i \cap G_i$ is closed in $X$. In other words $F_i$ is closed for all $i \in I$. Now

$$F \cap U = \bigcup_{i\in I}(F_i \cap U) = \bigcup_{j=1}^{n} F_j \cap U.$$

Here we have used the fact that $U$ intersect only finitely many of the sets $X_i$, so in particular only finitely many $F_i$.Above we denote these $F_i$ that intersect $U$ by $f_1, \ldots, F_n$. We have that

$$x \in \overline{F \cap U} = \overline{\bigcup_{j=1}^{n} F_j \cap U} = \bigcup_{j=1}^{n} \overline{F_j \cap U} \subset \bigcup_{j=1}^{n} \overline{F_j} = \bigcup_{j=1}^{n} F_j \subset F.$$

Here we have used Proposition 3.17(4) - finite union of closures is the closure of unions.

An example which shows that the assumption of local finiteness is essential - let $X_x = \{x\}$ be a singleton for every $x \in \mathbb{R}$. Then the collection $\mathbf{X} = (X_x)_{x\in\mathbb{R}}$ is a closed covering of $\mathbb{R}$. However the standard topology of $\mathbb{R}$ is not coherent with $\mathbf{X}$. Actually for every subset $U$ of $\mathbb{R}$ we have that $U \cap X_x$ is either an empty set or a singleton $X_x$, which is open in $X_x$, but $U$ need not to be open in $\mathbb{R}$. If this family would be coherent then the topology of $\mathbb{R}$ would be discrete.

6. Show that every open subset of $\mathbb{R}$ can be triangulated (as a topological space). (Hint: previous exercise might come in handy).

**Solution:** Suppose $U \subset \mathbb{R}$ is open. First we show that all connected components of $U$ are open intervals (possibly unbounded). This is seen as following. Suppose $\mathbf{x} \in U$ and let $C$ be a component of $\mathbf{x}$ in $U$. Since the connected subsets of $\mathbb{R}$ are intervals (Proposition 3.13(4)), $C$ is an interval. This interval cannot contain its endpoints. Indeed suppose $C$ contains its left end-point $c$. Then $c \in U$, so, since $U$ is open, there exists an interval $]a, b[$ such that

$$c \in ]a, b[ \subset U.$$

It is clear that the union $]a, b[ \cup C$ is an interval, hence connected. Moreover it contains $c$ and is bigger than $C$. But that contradicts the maximality of $C$, as a component. Hence $C$ must be an open interval.

Since every space is a disjoint union of its components, we have that

$$U = \bigcup_{i\in I} ]a_i, b_i[,$$

where union is disjoint (and for some $i$ we can have $a_i = -\infty$ or $b_i = \infty$). In fact it can be shown that this union is at most countable, but we do not need that fact).

For every $i \in I$ we can choose an increasing sequence

$$\ldots < a_i^{-n} < a_i^{-n+1} < \ldots < a_i^{-1} < a_i^0 < a_i^1 < \ldots < a_i^n < a_i^{n+1} < \ldots$$

**unlimited in both directions**, so that

$$\lim_{n\to\infty} a_i^{-n} = a_i,$$

$$\lim_{n\to\infty} a_i^n = b_i.$$

It is clear that

$$]a_i, b_i[= \bigcup_{j\in\mathbb{Z}} [a_i^j, a_i^{j+1}],$$

$$U = \bigcup_{i\in I, j\in\mathbb{Z}} ]a_i, b_i[.$$

The family $([a_i^j, a_i^{j+1}])_{i\in I, j\in\mathbb{Z}}$ is easily seen to be locally finite. Since this family is also a closed cover of $C$, the topology of $C$ is coherent with this family.

The collection $K = ([a_i^j, a_i^{j+1}])_{i\in I, j\in\mathbb{Z}}$ of 1-simplices and all their vertices is a simplicial complex - different simplices intersect, by construction, in their vertices only, at most. The polyhedron $|K|$ of this complex is precisely $U$. Also its weak topology is (by local finiteness of the cover $K$) the same as the standard topology of $U$. So this complex is a triangulation of $U$.