Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013

Exercises 1 (for the exercise session Tuesday 10.09)

1. Suppose $\mathbf{x}, \mathbf{y} \in V$, where V is a vector space and $\mathbf{x} \neq \mathbf{y}$. Prove that there exists unique line ℓ in V that contains both $\mathbf{x}$ and $\mathbf{y}$. Also show that in this case

$$\ell = \{(1-t)\mathbf{x} + t\mathbf{y} \mid t \in \mathbb{R}\} = \{\lambda\mathbf{x} + \mu\mathbf{y} \mid \lambda, \mu \in \mathbb{R}, \lambda + \mu = 1\}.$$

   We have defined a line to be a set of the form $\mathbf{y} + W$, where $W \subset V$ is a 1-dimensional vector subspace of V.

2. In the proof of the Lemma 2.4 we have shown that a given non-empty affine set A can be written in the form $A = \mathbf{v} + W$, where $\mathbf{v} \in A$ and W is a vector subspace of V. Complete the proof by showing that W is unique, while $\mathbf{v}$ can be chosen arbitrary from A.

3. Determine whether the set of vectors

$$A = \{(2, 1, -3), (6, 3, -4), (5, 2, -8), (9, 4, -9)\}$$

   in $\mathbb{R}^3$ is affinely independent or not. In case it is not also give an example of a point $\mathbf{x} \in \mathrm{conv}(A)$ that has two different representations as a convex combination of points of A (together with these combinations).

In the following exercises we need the concept of an "extreme point". Suppose C is a convex subset of a vector space V and $\mathbf{z} \in C$. We say that $\mathbf{z}$ is an **extreme point** of C if it is not an "interior point" of any proper closed interval of C i.e., precisely put if there do not exist $\mathbf{x}, \mathbf{y} \in C$ and $t \in]0, 1[$ such that $\mathbf{x} \neq \mathbf{y}$ and

$$\mathbf{z} = (1-t)\mathbf{x} + t\mathbf{y}.$$

4. In this exercise you are allowed to skip the proofs and calculations. The answer supported by drawings and 'intuition' is acceptable.
   (continues on the next page).

Determine the extreme points of the following convex sets:
I^2 (the square), $\overline{B}^n$ (closed unit ball), B^n (open unit ball), the closed upper half of the plane

$$H = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0\},$$

the closed quarter of the plane

$$F = \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0\}.$$

Also check for every of these sets if the statement
"the convex set is a convex hull of its extreme points" is true for them. Can you make a conjecture for which convex sets (in general) this statement is true?

5. Suppose the set $A = \{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is affinely independent subset of a vector space V. Let $\sigma = \text{conv}(A)$ be the simplex spanned by A. Show that a point of σ is an extreme point of σ if and only if it is one of the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$.
(This implies that the set of vertices of a simplex is determined by the simplex itself).

6. Deduce, using previous exercise, that the square I^2 and the closed ball $\overline{B}^2$ are not simplices, by showing that the corresponding sets of their extreme points are not affinely independent. You can use your answer to the exercise 4, where you have determined the sets of extreme points in question (in case you have done the exercise 4).

   Bonus points for the exercises: 25% - 2 point, 40% - 3 points, 50% - 4 points, 60% - 5 points, 75% - 6 points.