General theory of inverse problems: existence, uniqueness, stability and regularization

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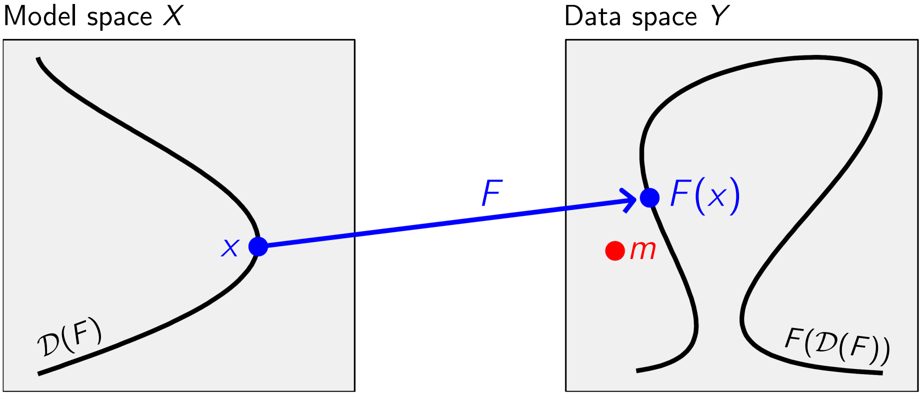
Inverse problem = interpretation of an indirect measurement modelled by a forward map F



Consider the measurement model $m = F(x) + \varepsilon$. We want to know x, but all we can do is measure m that depends indirectly on x.

The practical measurement m can be thought of as infinite-precision data $F(x)$ corrupted with additive noise ε.

Ill-posed inverse problems are defined as opposites of well-posed direct problems

Hadamard: a problem is *well-posed* if the following conditions hold:

1. A solution exists,
2. The solution is unique,
3. The dependence of the solution on the input is continuous.

Well-posed direct problem: input x, find infinite-precision data $F(x)$.

Ill-posed inverse problem: input noisy data $m = F(x) + \varepsilon$, recover x.

The solution of an inverse problem is a *set of instructions* for recovering x stably from m

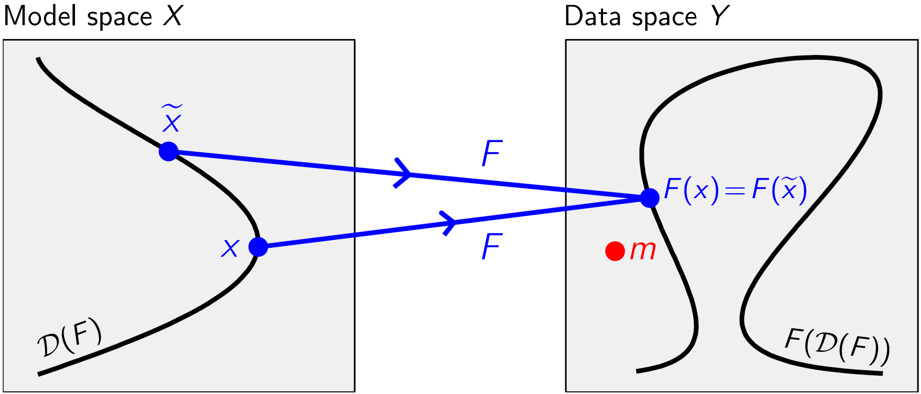
Those instructions need to be

(i) confirmed by rigorous mathematical analysis, and

(ii) implementable as an effective computational algorithm.

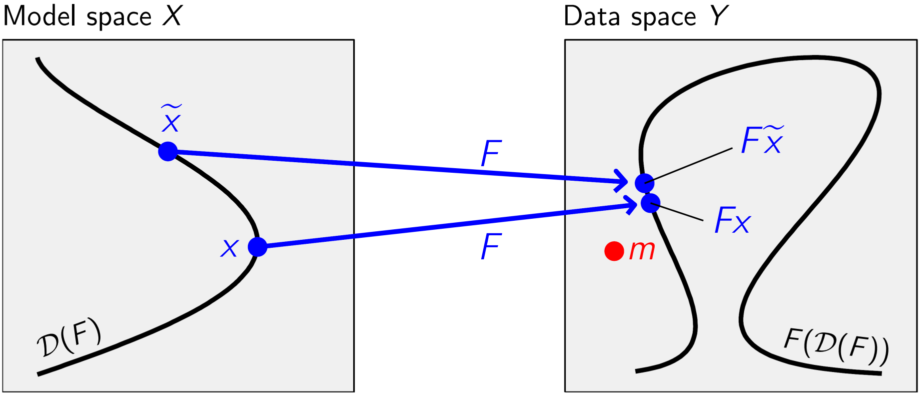
Since the forward map has no continuous inverse, it is impossible to recover x stably from m alone. The insufficient measurement data needs to be complemented by *a priori* knowledge.

Uniqueness: can two different objects produce the same infinite-precision data?



Ill-posedness means that the forward map F does not have a continuous inverse. Therefore, recovery of x from infinite-precision data $F(x)$ is unstable even if F is one-to-one. Furthermore, in general the data is not in the range: $m \notin F(\mathcal{D}(F))$.

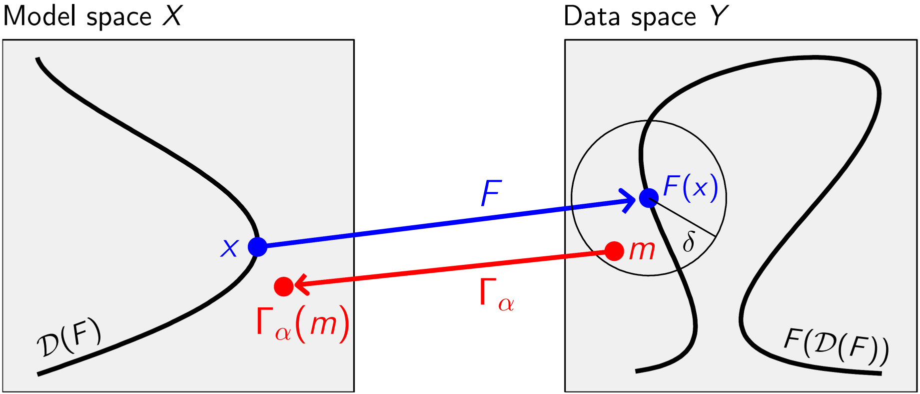
Conditional stability research studies the difference between images and preimages



Conditional stability results have the form $\|x - \widetilde{x}\|_X \leq f(\|Fx - F\widetilde{x}\|_Y)$, where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying $f(0) = 0$.

However, in general the data is not in the range: $m \notin F(\mathcal{D}(F))$, and the above inequality cannot be applied to m.

Regularization means constructing a continuous map $\Gamma_\alpha : Y \to X$ that inverts F approximately



The reconstruction $\Gamma_{\alpha(\delta)}(m)$ needs to approach x along a continuous path as the noise level $\delta \to 0$.

The solution of an inverse problem is to design and implement the map Γ_α so that it contains appropriate prior information.

A *regularization strategy* needs to be constructed so that the assumptions below are satisfied

A family $\Gamma_\alpha : Y \to X$ of continuous maps parameterized by $0 < \alpha < \infty$ is a *regularization strategy* for F if

$$\lim_{\alpha \to 0} \|\Gamma_\alpha(F(x)) - x\|_X = 0$$

for each fixed $x \in \mathcal{D}(F)$.

A regularization strategy with a choice $\alpha = \alpha(\delta)$ of regularization parameter is called *admissible* if

$$\alpha(\delta) \to 0 \text{ as } \delta \to 0,$$

and for any fixed $x \in \mathcal{D}(F)$ we have

$$\sup_{\|m - F(x)\|_Y \leq \delta} \left\{ \|\Gamma_{\alpha(\delta)}(m) - x\|_X \right\} \to 0$$

in the asymptotic limit $\delta \to 0$ of no noise.