Semigroups and Delay eq. Solution 8.

1. Let \( f \in \mathcal{C}_0, BC \subset L^p, 1 \leq p < \infty, \) and \( f(t) \in \mathcal{C}_0^*(\mathbb{R}) = \mathcal{W}_\infty, \) and define

\[
T(t)f := f * \mu_t, \quad (t \geq 0)
\]

Firstly, when is \( T(t) \) for a bounded operator?

**Lemma:** \( \int |f * \mu_t|^p dm(f, y) \leq \|f\|_{L^p}, 1 \leq p < \infty, \)

Proof: \( p = \infty \) is clear. For \( 1 \leq p < \infty \) we can assume \( \|\mu_t\|_1 = 1, \) and then use Jensen's inequality:

\[
\int |f(y) f(x-y)|^p dm(f, y) \leq \int \left| \int f(y) f(x-y) dm(f, y) \right|^p dm(f, y)
\]

By the definition:

\[
\int |f(y) f(x-y)|^p dm(f, y) = \|f\|_p^p \cdot \|\mu_t\|_1^p \quad \text{Done.}
\]

Lemma instantly implies \( T(t) \in \mathcal{B}(L^p) \) for all \( p. \)

It is also easy to see that \( T(t)BC \subset BC \) to \( T(t)BC(BC). \)

Finally one can check that \( T(t)C_0 \subset C_0, \) so that \( T(t)C_0 = T(t)C_\infty \subset C_0 \) too, and that \( T(t)C_0 \in C_0. \)
Next, when does $T(x+1) = T(x)T_T(x)$ hold?

Note that the Schwartz class $\mathcal{S}$ is a subset of $\mathcal{F}$ for each choice of $\mathcal{F}$. Therefore, if $T$ is a semigroup, then for all $x \in \mathcal{S}$ holds

\[
\psi(x + 1) = \psi(x) \psi_T(x)
\]

(i) $\psi(x + 1) = \psi(x) \psi_T(x)$

(ii) $\psi(x + 1) = \psi(x) \psi_T(x)$, $T > 0$.

Then if we assume measurability of mappings $x \mapsto \psi_T(x)$, for each fixed $x \in R$, it is well known that the fact that $\psi_T(x) = e^{\psi(x)}$ for some measurable $\psi : R \to R$.

But note that Fourier $\mathcal{F}$ of a finite measure is a bounded space. So $\sup \Re \psi(x) < \infty$.

Hence, a necessary and, for the family $S_T(x)$ in

\[
\psi(x) \preceq e^{\psi(x)}
\]

for a measurable $\psi : R \to R$ such that $\sup \Re \psi < \infty$.

But once we have this explicit candidate, it is easy to check that the above and, it is also sufficient:

\[
\psi(x + 1) = \psi(x) \psi_T(x) = (e^{\psi(x)} \cdot e^{\psi_T(x)}) = e^{\psi(x) + \psi_T(x)} = \psi(x + 1)
\]

Done.
Ex 2: When is \( T \) strongly compact on \( E \) ?

As always, \( T \), is enough to prove on a dense set. Schwartz class \( S \) is dense in \( L^1 \), \( p < \infty \), and on \( (C_0, \| \cdot \|_{\infty}) \), but not on \( L^\infty \) or \( BUC(\Omega) \).

**Case \( L^p \), \( 2 \leq p < \infty \):**

Riesz-Thorin Theorem implies that for \( p \in [1, 2] \) if \( M, t \),

\[ \| f \|_p \leq M \| f \|_p \quad (f \in L^p) . \]

Now let \( f \in S \) so that \( f^\ast u \in S \) too, meaning we can take its \( \mathcal{F} \)-transform \( \mathcal{F} f^\ast u \in S \) and apply \( \mathcal{F}^{-1} \) to it in the following way:

Let \( p \in [2, \infty] \)

\[ \| f^\ast u - f \|_p \leq M_2 \| \mathcal{F} f^\ast u - f \|_q \]

\[ = M_2 \| \mathcal{F} u - 1 \|_q \]

\[ = M_2 \| 1 e^{+iw} - 1 \|_q \quad (t \to 0). \]

Now, as \( t \to 0 \), \( 1 e^{+iw} - 1 \to 0 \) pointwise, and it bounded \( 1 e^{+iw} - 1 \leq e^{+\| REw \|_+} + 1 \). Since \( f \in L^p \), we can use \( \text{DCT} \) to infer

\[ \lim_{t \to 0} \| f^\ast u - f \|_p = 0, \]

proving strong compact on the set \( S \).
As $T$ is dense in spaces $L^p$, $p < \infty$ and $(C_0, || \cdot ||_\infty)$, we have proved

$T$ is S.C. on $L^p$, $p \in [2, \infty)$ and on $(C_0, || \cdot ||_\infty)$.

As for the spaces $L^\infty$ and $BUC$, I must admit
I don't know. For what it's worth, I say I suspect
strong cont. on $BUC$ but not on $L^\infty$.

**Cases $L^p$, $1 < p \leq 2$**

For $2 < p < \infty$, space $L^p$ is $\ast$-reflexive.

Then we know by general theory that

$(L^p)^{\ast} = (L^p)^\ast = L^q$,

and $T^\ast = T^\ast \ast$ is strongly cont. Since, it's easy to check, $T^\ast \ast$ acts like

$T^\ast g(x) = \int dy g(x + y)$ \hspace{1cm} (x \in \mathbb{R}, g \in L^q),

the original $T(g)$ must also be S.C. on $L^q$.

\[ \therefore T \text{ is S.C. on } L^p, \quad 1 < p \leq 2. \]

We also note that $(L^p)^\ast = L^q$, $1 < p < \infty$. 

Case $L^1$

By well known fundamental results, S.C. will follow from weak cont.

Let, then $f \in L^1$ and $g \in L^{1*} = L^\infty$. Then

\[
< f * m_k - f | g > = \int dx \, g(x) \int dy \, (f(x-y) - f(x))
\]

\[
= \int dy \, g(y) \int dx \, g(x) (f(x-y) - f(x))
\]

\[
g(\tau y f - f)
\]

where $g \in L^{1*}$ is viewed as a distribution acting like

\[g(x) = \int g(x,y) \, \delta(y)
\]

It's sufficiently clear that $y \to g(\tau y f - f) = : \delta f y$ is cont. and 0 at $y = 0$.

Now, since $\mu = e^{t\mu}$ pointwise, we expect that $\mu_k \to \mu$ in some sense. But for that we need to

\underline{assume the family $\mu_k$ is tight}, meaning that

\[\forall \varepsilon > 0 \exists K \in \text{compact such that } \mu_k(K \cap E) \leq \varepsilon \text{ for all } k \geq 0.
\]

This extra assumption then allows us to assume the frame $t$ is compactly supported (which it is not, but we may now assume that, for simplicity)
Then both \( \mu \) and also \( \phi \) have Fourier transforms which are functions and we can manipulate

\[
\int d\mu(y) \phi(y) = \int d\mu(y) \hat{\phi}^\dagger
\]

\[
= \int dp \ e^{iwp} \hat{\phi}(p)
\]

\[
(\text{DCT}) \quad \int dp \ \hat{\phi}^\dagger(p)
\]

\[
= \hat{\phi}(0)
\]

\[
= \phi(0) = 0.
\]

Thus \( T \) is weakly, and therefore strongly, continuous. Done.
Ex 8 \( x^c \) for \( C_0 \) and \( L^1 \):

\[ C_0^* = \mathcal{M} = \text{"finite measures"} \]

\[ \mathcal{N}BV(R) = \{ f \in BV(R) : \lim_{x \to \pm \infty} f(x) = 0 \} \]

I'm fairly confident that \( x^c = L^1 \)dm = AC,
but all I can really prove is that \( T^* \) is S.C. on \( L^1 \)dm.

\[ T^* \text{ is S.C. on } L^1 \]dm \( \equiv L^1 \).

We regard \( C_0^* \) as NBV so that \( T^* \) acts on functions:

\[ T^*(g) = g \ast \mu^+ \]

\[ = \left( \int_{\mathbb{R}} g(x+y) \, dy \right) \ast \mu^+ \quad (g \in \mathcal{N}BV) \]

It's quick to check that \( g \ast \mu^+ \in \mathcal{N}BV \) and that

\[ d(T^*(g)) = d(g \ast \mu^+) \]

\[ = (g \ast \mu^+) \, dx \]

If \( g \in AC \) ("\( dg = g \, dx \)"").

Therefore, since \( \| F \|_{NBV} = \| F' \|_{L^1} \) whenever \( F \in AC \),
we see

\[ \| T^*(g) - g \|_{NBV} = \| g \ast \mu^+ - g \|_{L^1} \rightarrow 0 \quad \text{as } t \to 0 \]

by the S.C. on \( L^1 \).

\[ \therefore T^* \text{ is S.C. } L^1 \equiv AC \]
Finally, I suspect \((L^1)^\circ = BUC\), but, once again, I fail to give rigorous justification. It seems, at least when \(W\) is not bounded, that \(T^*\) is not S.C. on \(L^0\setminus BUC\) but on \(BUC\) we indeed have:

\[ T^*_y \text{ and } T^z \in S.C. \text{ on } BUC. \]

Let \(f \in BUC(R)\). Then

\[ \|f - f\|_\infty = \sup_{y \in [0,1]} |\langle f, y - f \rangle| \]

\[ = \sup_{y \in [0,1]} \left| \int dx \, g(x) \left( f(x+y) - f(x) \right) \right| \]

(Fubini)

\[ = \sup_{y \in [0,1]} \left| \int dx \, g(x) \left( \frac{2y}{L} f(x) - f(x) \right) \right| \]

\[ \leq \int dy \, \|g\|_1 \|L_y f - f\|_\infty \]

\[ \leq \int dy \, \|L_y f - f\|_\infty \]

Now, translation is S.C. on \(BUC\) (contrast with \(L^0\) !), so that, by the arguments identical to above, we get

\[ \exists \lim_{y \to 0} \int dy \, \|L_y f - f\|_\infty = 0. \text{ Done.} \]
Ex 4: Basic key observation is to note that, for each $x$,

$$C_x = f \mapsto Tf(x)$$

defines a bounded linear functional on $C_0$. By Riesz, we know then that

$$Tf(x) = \mu_x(f) \quad (f \in C_0)$$

for some family $\mu_x \in \mathfrak{M}$ of measures.

Now, the fact that

$$T(G(y)f)(x) = (G(y)Tf)(x) = (Tf)(x + y)$$

translates into

$$\mu \circ G(y) = \mu_{x+y} \quad (x, y \in \mathbb{R})$$

In particular, if $x=0$, $\mu \circ G(y) = \mu_y$, i.e.

$$(Tf)(x) = \mu_x(f)$$

$$= \mu_x(G(x)f)$$

$$= (\delta_{x+y}(f) \mu_{x+y})$$

$$= \tilde{\mu}_{x+y}(f), \quad \text{where } \tilde{\mu}(E) = \mu(E)^x.$$  Done.