Ex 1: By continuity of the mappings $t \mapsto f(t)$ and $t \mapsto A(f(t))$ the integrals can be taken in the sense of Riemann. By definition, then, we have the existence of

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(t_k) \Delta t_k = \lim_{N \to \infty} A \left( \sum_{k=1}^{N} f(t_k) \Delta t_k \right).$$

But now observe that we also have the existence

$$\exists \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(t_k) \Delta t_k = \int_{a}^{b} f(t) \, dt,$$

so that, by the very definition of closedness, we can deduce that $\int_{a}^{b} f(t) \, dt \in \mathcal{D}(A)$ and

$$A \left( \int_{a}^{b} f(t) \, dt \right) = \int_{a}^{b} A(f(t)) \, dt.$$

\[ \square \]

Alternative way: We can assume $\mathcal{D}(A)$ is dense; otherwise take $X = \mathcal{D}(A)$. Then, since $A$ is also closed, we know the adjoint $A^*$ is densely defined. For every $x^* \in \mathcal{D}(A^*)$

$$\langle A \int_{a}^{b} f(t) \, dt, x^* \rangle = \langle \int_{a}^{b} f(t), A^* x^* \rangle = \int_{a}^{b} \langle f(t), A^* x^* \rangle dt = \int_{a}^{b} \langle A^* f(t), x^* \rangle dt = \langle \int_{a}^{b} A^* f(t), x^* \rangle.$$  

Okay, I forgot to check $\int_{a}^{b} f \in \mathcal{D}(A)$, but in our "semigroup-case" it's known. If you accept that, then the above implies $\int_{a}^{b} f = \int_{a}^{b} A f$ by denseness of $\mathcal{D}(A^*)$. "Done."
Ex. 2: (i) Since \( F \) is complete, it is sufficient to show \([A_n x]_n\) is Cauchy for all \( x \in X\).

Given \( \varepsilon > 0 \), choose \( y \in E \) such that \( \|x - y\| < \varepsilon \).

Then estimate

\[
\|A_n x - A_k x\| \leq \|A_n (x - y)\| + \|A_n y - A_k y\| + \|A_k (y - x)\|
\]

\[
\leq 2M \|x - y\| + \|A_n y - A_k y\|
\]

by uniform boundedness of \( A_n \). Now, since \([A_n y]_n\) is Cauchy, we find \( N_{x,y} \) such that for \( m, k \geq N_{x,y} \),

\[\|A_n y - A_k y\| < \varepsilon.\]

This finally gives

\[
\|A_n x - A_k x\| < (2M + 1) \varepsilon, \quad (m, k \geq N_{x,y}).
\]

(ii) It clear each extension \( A \) is linear. Boundedness follows from

\[\|A_n x\| = \|A_n x\| \leq M \|x\|, \quad (x \in X).
\]

Remark: Completeness of only \( F \) is required.
Suppose $R(\lambda_0 - A) = \mathbb{X}$ for some $\lambda_0 > 0$.

Now, given $\lambda > 0$ and any $y \in \mathbb{X}$, let us see if
we can find $x \in D(A)$ such that

$$(\lambda - A)x = y.$$ 

It feels like a good idea to add and subtract $\lambda_0$:

$$(\lambda - \lambda_0)x + (\lambda_0 - A)x = y.$$ 

We can apply $(\lambda_0 - A)^{-1}$ to both sides, resulting

$$[(\lambda - \lambda_0)(\lambda_0 - A)^{-1}]x = (\lambda_0 - A)^{-1}y.$$ 

But $(\lambda_0 - A)^{-1}$ is a bounded operator of norm
less than or equal to $|\lambda_0 - \lambda|/\lambda_0$ (by the dissipativity).

Therefore, if $|\lambda_0 - \lambda| < \lambda_0$, we can define the inverse

$$(\lambda - \lambda_0)(\lambda_0 - A)^{-1}$$ 

by the geometric series

$$\sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n.$$

Thus, finally,

$$x = \sum_{n=0}^{\infty} \frac{(\lambda_0 - \lambda)^n}{(\lambda_0 - A)^{n+1}} y,$$

and $R(\lambda - A) = \mathbb{X}$ for all $\lambda \in (0, 2\lambda_0)$. Iteration then gives
the same for all $\lambda \in (0, \infty)$.
Ex. 4: That \( \text{ker}(A) = \{ f \in C^1[a, b] : f(a) = 0 \} \)

is not dense in \((C^1[a, b], \| \cdot \|_\infty)\) is clear, since for any \( f \in C^1[a, b] \) such that \( f(0) \neq 0 \) we see that its neighborhood \([g \in C^1 : \| g - f \|_\infty < |f(0)|] \) is disjoint from \( \text{ker}(A) \).

Now, given \( \lambda > 0 \) and \( f \in \text{ker}(A) \) we must prove

\[
\| \lambda f \|_\infty = \| \lambda f + f' \|_\infty \geq \lambda \| f \|_\infty.
\]

The crucial observation is that \( \| f \|_\infty \) indeed attains its maximum \( \| f \|_\infty \). That is, there exists \( x_0 \in [0, 1] \) such that \( |f(x_0)| = \| f \|_\infty \).

The next thing is to note that \( f(x_0) \) must be zero unless \( x_0 = 1 \), in which case it must, by maximality of \( |f(1)| \), still be true that \( |\lambda f(1) + f'(1)| \geq \lambda |f(1)| \).

(You can assume, by notation, that \( f(1) > 0 \); then it should be apparent...)

So, in any case we have

\[
\| (\lambda + 1) f \|_\infty \geq |\lambda f(x_0) + f'(x_0)|
\]

\[
= \lambda |f(x_0)|
\]

\[
= \lambda \| f \|_\infty.
\]
Ex 5: Let \( X = \text{BUC}(\mathbb{R}) \) or \( L^p(\mathbb{R}) \)
and look at the differential eq. for some given \( g \in X \)
\[
\lambda f + f' = g \quad \|e^{\lambda x}\|_{L^1} = 1
\]
\[
\Rightarrow \int e^{\lambda x} f(x) \, dx = e^{\lambda x} g(x) \quad \|\int_0^x\|
\]
\[
\Rightarrow e^{\lambda x} f(x) - e^{\lambda x} f(0) = \int_0^x e^{\lambda y} g(y) \, dy, \quad (\forall x, \alpha \in \mathbb{R}).
\]

This much we can always do. First question is what to do with the term \( e^{\lambda x} f(0) \) ? If we can somehow "handle it," the second question is whether resulting \( f \) belongs to \( X \).

If \( \text{Re} \lambda > 0 \) or \( \text{Re} \lambda < 0 \), we can let \( \alpha \to -\infty \) or \( \infty \) respectively, thus obtaining

\[
f(x) = \begin{cases} 
\int_{-\infty}^{x} e^{\lambda y} g(y) \, dy, & \text{if } \lambda > 0, \\
-\int_{x}^{\infty} e^{\lambda y} g(y) \, dy, & \text{if } \lambda < 0.
\end{cases}
\]

\[
= (\theta(x) e^{\lambda x} f(0)), \quad \text{or } -\theta(x) e^{\lambda x} f(0),
\]

where \( \theta = \chi_{[\alpha, \infty)} \) and \( \hat{\theta} = \chi_{(-\infty, \alpha]} \). Now, it is a general fact that for all \( p \in [1, \infty] \)
\[
\|xf\|_{L^p} \leq \|f\|_{L^1} \|x\|_{L^p}, \quad (x \in L^1, \text{ } f \in L^p).
\]

(Proof by Jensen: Assuming \( \frac{1}{p} + \frac{1}{q} = 1 \)
\[
\int \int |x f(x, y) (x - y)|^p \leq \int \int |x f(x, y) (x - y)|^p = \|f\|_{L^p} \|x\|_{L^1}.
\)
Ex 5: Applying it to our case with \( y = e^{-\lambda t} \) or \( e^{\lambda t} \), so that \( \|x + y\| = \|\text{Re}\lambda\| \),
we get the bound
\[
\|f\|_p \leq \|\text{Re}\lambda\| \|g\|_p.
\]
Thus it should be clear that in all cases of \( \mathcal{E} \) we have \( \mathcal{R}(\lambda - \lambda) = \mathcal{E} \) and \( (\lambda - \lambda)^f \) is bounded.
Since this holds true for all \( \text{Re}\lambda \neq 0 \) we have \( \omega(\lambda) \leq \mathcal{R} \).

**Case \( \mathcal{E} = \text{BUC}(\mathcal{R}) \), \( \lambda \in \mathbb{R} \):**

In this case \( \lambda \in \text{Re}(\text{BUC}) \) for
\[
\lambda f + f' = 0 \quad (\Rightarrow \quad f(x) = (c_0) e^{\lambda x} \in \text{BUC}(\mathcal{R})),
\]
for arbitrary \( (c_0) \in \mathbb{C} \).

The same reasoning shows that \( \lambda f + f' = 0 \) has no solutions when \( \mathcal{E} = L^p \), for \( \lambda \neq 0 \) and \( L^p \), \( p < \infty \).

**Case \( \mathcal{E} = L^p(\mathcal{R}) \), \( \lambda \in \mathbb{R} \):**

Suppose \( g \in \mathcal{R}(\lambda - \lambda) \), so that \( \mathcal{E}f \in \mathcal{E} \) such that
\[
\lambda f + f' = g.
\]
As before, we have that for all \( \lambda, a \) holds.
Ex 5: \( \int e^x \, dx - \int a e^{\frac{x}{a}} \, dx = \int_{-\infty}^{\infty} e^y g(y) \, dy \).

Now, since \( f \in O(\lambda) \), \( f \) is cont. and remains at \( +\infty \). This implies \( g \) must satisfy

\[
\lim_{x \to \infty} \int_{-\infty}^{\infty} e^{-y} g(y) \, dy = \hat{g}(i\lambda) = 0.
\]

Subcase \( p = 1 \):

When \( g \in L^1 \) the \( \hat{g} \) is actually pointwise well-defined. In fact we have \( \| \hat{g} \|_{L^1} \leq \| g \|_{L_1} \), so that for fixed \( \lambda \)

\[ L^1 : g \mapsto \hat{g}(i\lambda) \] 

is cont. Hence \( \{ g \in L^1 : \hat{g}(i\lambda) = 0 \} \) is a closed proper subspace which contains \( \Re(a - \lambda) \).

\[ \therefore \ iR = \Re(\Lambda_{L^1}) \]

Subcase \( p \in (1, \infty) \):

We show that actually \( \Re(a - \lambda)^{-p} \in L^p_c \) ("compactly supported \( L^p \)"), so that it is dense.

Let \( g \in L^p_c \) such that \( \sup_{x} g \in [\alpha, \beta] \), and

\[ \int_{-\infty}^{\infty} e^{-y} g(y) = \int_{\alpha}^{\beta} e^{-y} g(y) = A \in C. \]

Note that when \( \lambda \to 0 \) \( g \) does not satisfy the necessary cond. to be in the range of \( a - \lambda \).
On the other hand, if \( A \) is zero then \( x \mapsto \int_{-\infty}^{x} e^{uy} y^g \) is compactly supported and thus in \( L^p \) too, so that \( g \in \mathcal{R}(A^*) \).

Our job, then, is to "modify" \( g \) a tiny bit with respect to \( L^p \)-norm such that \( A = 0 \). Define, for \( \epsilon > 0 \),

\[
\epsilon \xi := -\epsilon e^{-\frac{1}{2}} \arg A \times \left[ 0, 1 \right], \epsilon \xi \in L^p.
\]

Then, if \( M = \max \left[ |A\xi|, 1 \right] \),

\[
\int_{-\infty}^{\infty} \left( e^{uy} y^g + \epsilon \xi \right) = \int_{-\infty}^{\infty} e^{uy} y^g + \int_{-\infty}^{\infty} \epsilon \xi = 0.
\]

This, by the reasoning above, implies \( g + e^{-\frac{1}{2}} \epsilon \xi \in \mathcal{R}(A^*) \).

Finally,

\[
\| g - (g + e^{-\frac{1}{2}} \epsilon \xi) \|_p = \| \epsilon \xi \|_p = \epsilon^{p-1/2},
\]

which, because \( p > 1 \), can be made as small as we please. Done.

\[ \therefore \mathbb{R} = \mathcal{C}(A^*). \]

Finally, either by Gille-Yosida (since \( \| (x-t)^g \| \leq \frac{1}{\sqrt{x}}, x > 0 \)) or by Lumer-Phillips (since \( A \) is on-dissipative) A generates a strongly cont. contraction semigroup.