whose root edge connects u_i to its parent, $i=1,\dots,M$. Then

$$\mu_N(\mathcal{E}(\tau_0)) = Z_N^{-1}\, w(\tau_0) \sum_{N_1+\dots+N_M = N+M-|\tau_0|} \prod_{i=1}^{M} Z_{N_i}$$

$$= w(\tau_0)\, Z_N^{-1} \sum_{i_0=1}^{M} \sum_{\substack{N_1+\dots+N_M=N+M-|\tau_0| \\ N_{i_0} \geq \frac{N+M-|\tau_0|}{M},\ N_j \leq A,\ j\neq i_0}} \prod_{i=1}^{M} Z_{N_i}$$

$$+ \quad R(A, N, \tau_0) \qquad (3.3)$$

where $A>0$ is fixed and the remainder $R(A,N,\tau_0)$ can be estimated as follows:

$$R(A,N,\tau_0) \leq M^2 w(\tau_0) \sum_{\substack{N_1+\dots+N_M = N+M-|\tau_0| \\ N_1 \geq \frac{N+M-|\tau_0|}{M},\ N_2\geq A}} Z_N^{-1} \prod_{i=1}^{M} Z_{N_i}$$

$$= M^2 w(\tau_0)\, \xi_0^{|\tau_0|-M} \sum_{-\,''\,-} \frac{Z_{N_1}\xi_0^{N_1}}{Z_N\xi_0^N} \left(\prod_{i=3}^{M} Z_{N_i}\xi_0^{N_i}\right)\left(Z_{N_2}\xi_0^{N_2}\right)$$

$$\leq M^2 w(\tau_0)\, \xi_0^{|\tau_0|-M}\, C M^{3/2} \sum_{N_2>A} N_2^{-3/2} \cdot Z_0^{M-2}$$

$$\leq C_{\tau_0} \cdot A^{-\frac{1}{2}}. \qquad (3.4)$$

On the other hand,

$$Z_N^{-1} \sum_{\substack{N_1+\dots+N_M = N+M-|\tau_0| \\ N_{i_0}\geq \frac{N+M-|\tau_0|}{M},\ N_j\leq A \text{ for } j\neq i_0}} \prod_{i=1}^{M} Z_{N_i}$$

$$= \xi_0^{|\tau_0|-M} \sum_{-n-} \frac{Z_{N_{i_0}} \xi_0^{N_{i_0}}}{Z_N \xi_0^N} \prod_{i \neq i_0} (Z_{N_i} \xi_0^{N_i})$$

$$\longrightarrow \xi^{|\tau_0|-M} \left(\sum_{k=1}^{A} Z_k \xi_0^k \right)^{M-1} \qquad (3.5)$$

as $N \to \infty$, which together with (3.4) proves the claim.

The existence of the limits (3.2) together with a precompactness property of $\{\mu_N\}$, which we omit here, implies by standard results (see e.g. Billingsley) that the limit μ exists, and $\mu(\mathscr{C}(\tau_0))$ is given by (3.2).

Single spine: Given $A > 0$ and τ_0 as before let

$$\mathscr{C}_A(\tau_0) = \{ \tau \mid B^\tau(R, v_0) = \tau_0 \text{ and at least two branches of } \tau \text{ originating at } \partial B^\tau(R, v_0) \text{ have size} > A \}.$$

Then $\mathscr{C}_A(\tau_0)$ is an open subset of $\mathscr{T}$ and (3.4) implies that

$$\liminf_{N \to \infty} \mu_N(\mathscr{C}_A(\tau_0)) \leq C_{\tau_0} \cdot A^{-\frac{1}{2}}$$

and hence $\mu(\mathscr{C}_A(\tau_0)) \leq C_{\tau_0} A^{-\frac{1}{2}}$ and

$$\mu\left(\bigcap_{A>0} \mathscr{C}_A(\tau_0) \right) = 0$$

But $\mathscr{T}_\infty \setminus \mathscr{S} = \bigcup_{\tau_0 \text{ finite}} \left(\bigcap_{A>0} \mathscr{C}_A(\tau_0) \right)$ and clearly $\mu(\mathscr{T} \setminus \mathscr{T}_\infty) = 0$.

<u>Distribution of branches</u>: Given $R > 0$ and a finite tree τ_1 with a marked leaf w different from v_0 and consider the simple path $(v_0 = s_0)\, s_1\, s_2 \dots (s_n = w)$ as a finite spine of τ_1. For $\tau \in \mathcal{S}$ let $\tau^{(n)}$ be the subtree spanned by $s_0, \dots, s_n$ and the branches rooted at $s_1, \dots, s_{n-1}$ with s_n marked. Define

$$\mathcal{S}_n(\tau_1) = \{\tau \in \mathcal{S} \mid \tau^{(n)} = \tau_1\}.$$

Then formula (3.5) implies that

$$\mu(\mathcal{S}_n(\tau_1)) = \xi_0^{|\tau_1|-1} \prod_{v \in \tau_1 \setminus \{v_0, w\}} w_{\sigma_v}$$

which is equivalent to the content of 1) and 2) of Thm. 3.2. $\square$

Hausdorff dimension of $(\mathcal{S}, \mu)$

<u>Lemma 3.2</u> For a critical G-W tree we have

i) $\langle |B(R, v_0)| \rangle_p = R$

ii) $P(\{\tau \in \mathcal{T} \mid h(\tau) > R\}) = \frac{2}{g''(1) R} + O(R^{-2})$

<u>Proof.</u> i) Let $H_i = \langle d_i(\tau) \rangle_p$. Then

$$H_0 = H_1 = 1,$$

$$H_{i+1} = \xi_0 \sum_{n=1}^{\infty} (n-1) w_n Z_0^{n-2} H_i$$

$$= \sum_{n=0}^{\infty} n p_n \cdot H_i = H_i = 1 .$$

Hence

$$\langle |B(R, v_0)| \rangle_p = \sum_{i=1}^{R} H_i = R .$$

ii) Is a classical result of Kolmogoroff (1931) and can be found in Athreya & Ney.

<u>Theorem 3.3</u> For the generic random tree $(\mathcal{S}, \mu)$ we have

$$\bar{d}_h = 2 \quad \text{and} \quad d_h = 2 \quad \text{almost surely.}$$

<u>Proof of $\bar{d}_h = 2$</u> For $\tau \in \mathcal{S}$ the contribution to $|B^\tau(R, v_0)|$ from a branch T rooted at s_i is $|B^T(R-i, s_i)|$. Since branches are indep. this gives

$$\langle |B(R, v_0)| \rangle_\mu = \sum_{i=1}^{R-1} (R-i) \sum_{\underline{k}', \underline{k}''} \prod_{j=1}^{R} p_{k_j' + k_j'' + 1} (k_j' + k_j'') + R$$

$$= \tfrac{1}{2} g''(1)\, R(R-1) + R .$$

That $d_h = 2$ a.s. follows by a closer analysis of the generating function g (see e.g. DJW 2). $\square$

<u>Theorem 3.4</u> For the generic random tree $(\mathcal{S}, \mu)$ we have

$$\bar{d}_s = \frac{4}{3} \quad \text{and} \quad d_s = \frac{4}{3} \quad \text{a. s.}$$

Proof of $\bar{d}_s \leq \frac{4}{3}$ and $d_s \geq \frac{4}{3}$ a.s.

That $d_s \geq \frac{4}{3}$ a.s. is a consequence of Thm. 1.1 and Thm. 3.3.
To show $\bar{d}_s \leq \frac{4}{3}$ we need two simple lemmas.

Lemma 3.3 For any finite tree $T \in \mathcal{T} \setminus \mathcal{T}_\infty$ and $0 < x \leq 1$ we have

$$P_T(x) \geq 1 - |T|x .$$

Proof. Let $T_1, \dots, T_{n-1}$ be the subtree of T rooted at the vertex v_i next to the root. By (a generalization) of Lemma 1.2 we have

$$P_T(x) = \frac{1-x}{n - \sum_{i=1}^{n-1} P_{T_i}(x)} .$$

If $|T| = 1$ then $P_T(x) = 1-x$, and the lemma follows by induction on $|T|$. $\square$

Lemma 3.4 Let $\tau \in \mathcal{S}$. For all $L \geq 1$ and $0 < x \leq 1$ we have

$$P_\tau(x) \geq 1 - \frac{1}{L} - Lx - \sum_{T \subset \tau}^{\leq L} (1 - P_T(x)),$$

where $\sum_{T \subset \tau}^{\leq L}$ indicates the sum over all branches of τ rooted at vertices $s_1, \dots, s_L$ on the spine.

Proof. Let $P_\tau^L(x)$ denote the contribution to $P_\tau(x)$ from paths ω not hitting s_{L+1}. It suffices to show the inequality for $P_\tau^L(x)$.

For $L=1$ it obviously holds since $P_\tau^1(x) \geq 0$. Assume it holds for $L-1$ and use Lemma 1.2 to write

$$P_L(x) = \frac{1-x}{n - P_{\tau_1}^{L-1}(x) - \sum_{k=1}^{n-2} P_{T_k}(x)}$$

where $T_1, \ldots, T_{n-2}$ denote the finite branches of τ rooted at s_1 and τ_1 the infinite branch and $n = \sigma_{s_1}$. Together with the induction hypothesis this gives

$$P_L(x) = \frac{1-x}{1 + (1-P_{\tau_1}^{L-1}(x)) + \sum_{k=1}^{n-2}(1-P_{T_k}(x))}$$

$$\geq \frac{1-x}{1 + \frac{1}{L-1} + (L-1)x + \sum_{T \subset \tau}^{\leq L}(1-P_T(x))}$$

$$\geq \frac{L-1}{L} \frac{1-x}{1 + (L-1)x + \sum_{T\subset\tau}^{\leq L}(1-P_T(x))}$$

$$\geq \left(1-\frac{1}{L}\right)(1-x)\left(1-(L-1)x - \sum_{T\subset\tau}^{\leq L}(1-P_T(x))\right)$$

$$\geq \left(1-\frac{1}{L}\right)(1-x) - (L-1)x - \sum_{T\subset\tau}^{\leq L}(1-P_T(x))$$

$$\geq 1 - \frac{1}{L} - Lx - \sum_{T\subset\tau}^{\leq L}(1-P_T(x)).$$

$\square$

Back to the proof of $\bar{d}_s \leq \frac{4}{3}$:

Let $s \neq r$ be any vertex on the spine having k left branches and ℓ right branches. The probability c_R that at least one of those branches has height $> R$ fulfills

$$c_R \leq (k+\ell)\left(\frac{2}{g''(1)R} + O(R^{-2})\right)$$

by Lemma 3.2 i). Hence the μ-probability q_R that a branch at s has height $> R$ fulfills

$$q_r \leq \left(\frac{2}{g''(1)R} + O(R^{-2})\right) \sum_{\ell,k\geq 0} (k+\ell)\varphi(\ell,k)$$

where

$$\varphi(\ell,k) = w_{\ell+k+2} Z_0^{\ell+k} \cdot \xi_0 = p_{\ell+k+1}$$

by Thm. 3.2. It follows that the last sum is $g''(1)$ such that

$$q_r \leq \frac{2}{R} + O(R^{-2}).$$

Using independence of branches one gets that the probability of the event $\mathcal{A}_R$ that all brances rooted at spine vertices $s_1, \ldots, s_R$ have height $\leq R$ fulfills

$$\mu(\mathcal{A}_R) = (1-q_R)^R \geq a > 0,$$

where a is indep. of R. Denoting by $\langle\cdot\rangle_R$ the expectation value of μ conditioned on $\mathcal{A}_R$ this gives

$$\langle Q_\tau(x)\rangle_\mu \geq a^{-1} \langle (1-P_\tau(x))^{-1}\rangle_R$$

$$\geq a^{-1} \left\langle \left(\frac{1}{R} + Rx + \sum_{T\subset\tau}^{\leq R} (1-P_T(x))\right)^{-1} \right\rangle_R$$

$$\geq a^{-1} \left(\frac{1}{R} + Rx + \left\langle \sum_{T\subset\tau}^{\leq R} (1-P_T(x))\right\rangle_R\right)^{-1}$$

$$\geq a^{-1} \left(\frac{1}{R} + Rx + \left\langle \sum_{T\subset\tau}^{\leq R} |T|\right\rangle_R x\right)^{-1}.$$

Since T in the last average has height $\leq R$ it follows that

$$\left\langle \sum_{T\subset\tau}^{\leq R} |T| \right\rangle_R \leq (1-q_R)^{-1} \sum_{i=1}^{R} \langle |B_\tau^i(R,s_i)|\rangle_\mu$$

where $B_\tau^i(R,s_i)$ denotes the subtree of τ spanned by the vertices in the branches rooted at s_i at distance $\leq R$ from s_i. Using independence of branches we get that $\langle |B_\tau^i(R,s_i)|\rangle_\mu$ is indep. of i and given by

$$\langle |B_\tau^i(R,s_i)|\rangle_\mu = \sum_{k,\ell\geq 0} (k+\ell)\varphi(k,\ell)\langle B_T(R,v_0)\rangle_p$$

$$= g''(1)R.$$

We conclude that, since $q_R \to 0$ as $R\to\infty$, we for R large:

$$\langle Q_\tau(x)\rangle_\mu \geq c'\left(\frac{1}{R} + Rx + g''(1)\cdot x\cdot R^2\right)^{-1}.$$

Setting $R = [x^{-1/3}]$ yields $\langle Q_\tau(x)\rangle_\mu \geq \underline{c}\, x^{-\frac{1}{3}}$ and hence $\bar{d}_s \leq \frac{4}{3}$. $\square$

<u>Remark</u> on the two-point function, mass and mass exponent for generic random trees.