$$p^{(2)}(x) \leq e^{-c' x^{-\varepsilon}}$$

while

$$P_L^{(1)}(x) = 1 - \frac{x^{1/4}}{\sqrt{L(x)}} + O(\sqrt{x})$$

$$\leq 1 - c'' x^{1/4 + \varepsilon/2} + O(\sqrt{x}).$$

It follows that

$$Q_C(x) \leq C''' x^{-1/4 - \varepsilon/2}$$

for all $C \in \mathcal{A}_\varepsilon$. Consequently

$$\langle Q_C(x) \rangle_\mu \leq cst. \, x^{-\frac{1}{4} - \varepsilon/2}$$

for all $\varepsilon > 0$. This gives $\alpha < \frac{1}{4} + \frac{\varepsilon}{2}$, i.e. $\alpha \leq \frac{1}{4}$ and hence $\bar{d}_s \geq \frac{3}{2}$. Choosing $N(x)$ more carefully one obtaines the claimed bounds by Borel-Cantelli. ▯

Fore combs with finite teeth we have

<u>Theorem 2.2</u> Let $a > 1$ and $p_\ell = c_a \ell^{-a}$, where c_a is a normalization factor such that $\sum_{\ell=1}^{\infty} p_\ell = 1$. For the random comb with tooth lengths independently distributed according to $(p_\ell)_{\ell \geq 1}$ we have $\bar{d}_h = \begin{cases} 3-a, & 1<a\leq 2 \\ 1, & a>2 \end{cases}$ while

$$\bar{d}_s = \begin{cases} \frac{4-a}{2} & \text{if } 1 < a \leq 2 \\ 1 & \text{if } a > 2. \end{cases}$$

3. Random infinite trees

3.1 Galton-Watson trees

A planar tree $\tau = (V, E)$ is a tree with

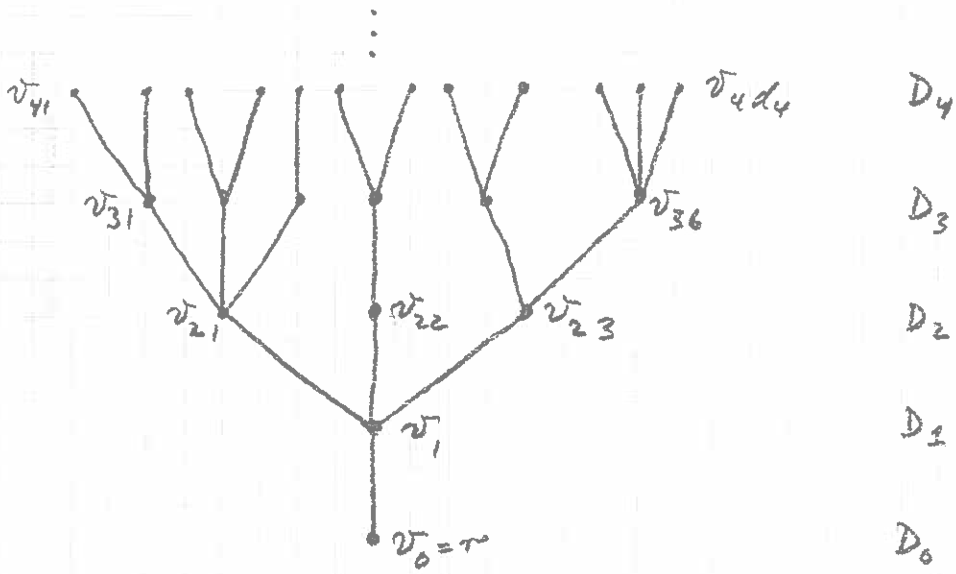
$$V = \bigcup_{n=0}^{\infty} D_i \quad , \qquad D_0 = \{x_0\} \quad \text{root}$$

$$D_i = \{v_{i1} < \ldots < v_{id_i}\} = i\text{'th height class}$$

$$E = \bigcup_{n=1}^{\infty} E_i$$

$$E_i = \{(v_{i1} v_{i-1 j_1}), \ldots, (v_{id_i} v_{i-1 j_{d_i}})\}, \; j_1 \le j_2 \le \ldots \le j_{d_i}$$

Assume also $\quad D_1 = \{v_{11}\} = \{v_1\}$, $(v_0 v_1) = \text{root edge}$



Two planar trees are identified if they are isomorphic by an isomorphism preserving the root and ordering of height classes.

$$h = \text{height of } \tau = \sup\{i \mid D_i \neq \emptyset\}$$

$$|\tau| = \text{size of } \tau = \sum_{i=1}^{\infty} d_i(\tau)$$

$$\mathcal{T} = \{\text{planar trees}\} = \Big(\bigcup_{N=1}^{\infty} \mathcal{T}_N\Big) \cup \mathcal{T}_\infty$$

$$\mathcal{T}_N = \{\tau \mid |\tau| = N\}, \quad N = 1, 2, \ldots, \infty .$$

Let w_n, $n \geq 1$, be given <u>branching weights</u>, $w_1 > 0$, $w_n > 0$ for some $n \geq 3$ and set

$$Z_N = \sum_{\tau \in \mathcal{T}_N} \prod_{v \in \tau \setminus r} w_{\sigma_v} \equiv \sum_{\tau \in \mathcal{T}_N} w(\tau). \quad (3.1)$$

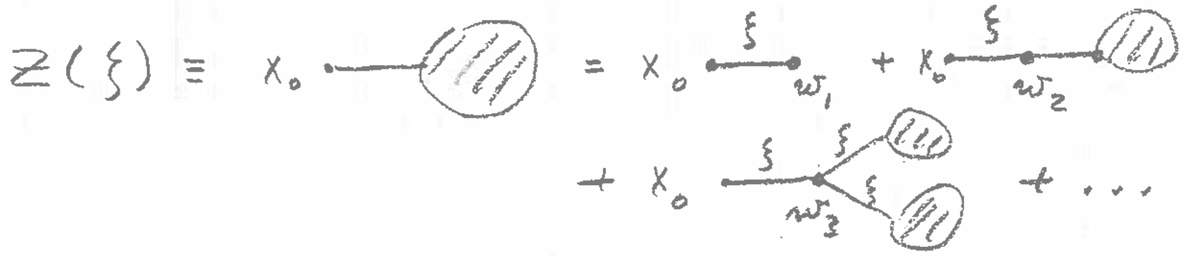
Generating functions:

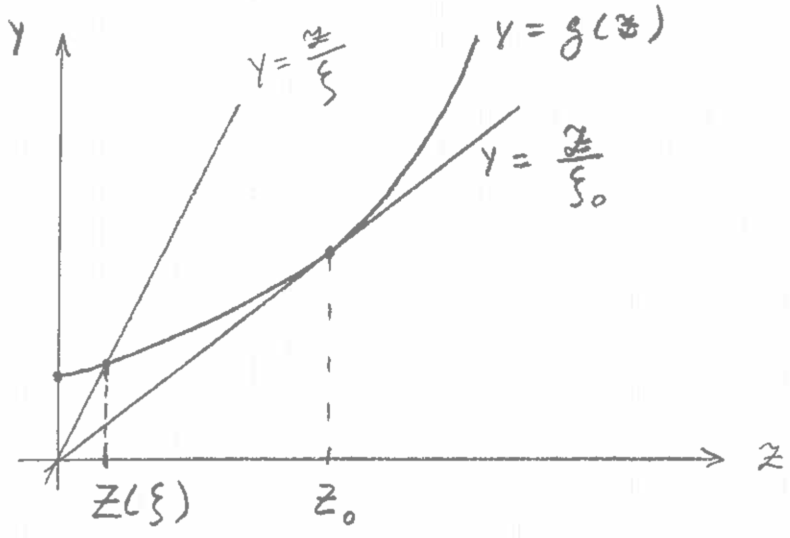
$$g(z) = \sum_{n=1}^{\infty} w_n z^{n-1}, \qquad |z| < \rho \quad (\rho > 0)$$

$$Z(\xi) = \sum_{N=1}^{\infty} Z_N \xi^N, \qquad |\xi| < \xi_0 .$$

Functional eq. for Z:

$$\boxed{Z(\xi) = \xi\, g(Z(\xi))} \qquad (I)$$





Genericity assumption: $\boxed{Z_0 < \rho}$ (II)

Holds, in particular, if $\rho = +\infty$, and $Z_0 \equiv Z(\xi$
is determined by

$$\boxed{Z_0\, g'(Z_0) = g(Z_0)} \qquad (\text{III})$$

Taylor expansion of g around Z_0 in (I) and using (III) gives

$$Z(\xi) = Z_0 - \sqrt{\frac{2g'(Z_0)}{\xi_0 g''(Z_0)}}\,(\xi_0 - \xi)^{1/2} + O(\xi_0 - \xi).$$

By standard transfer theorems (see e.g. Flajolet & Sedgewick IV.5 and VII.2) one gets

<u>Lemma 3.1</u> If $g(z)$ is aperiodic the asymptotic behaviour of Z_N is given by

$$Z_N = \sqrt{\frac{g'(Z_0)}{2\pi g''(Z_0)}}\; N^{-\frac{3}{2}}\, \xi_0^{-N}\left(1 + O(N^{-1})\right)$$

as $N \to \infty$. (A similar formula holds for periodic g)

<u>Ex. 3.1</u> For $w_n = 1$, $n \geq 1$, we have

$$g(z) = \frac{1}{1-z}, \qquad \rho = 1$$

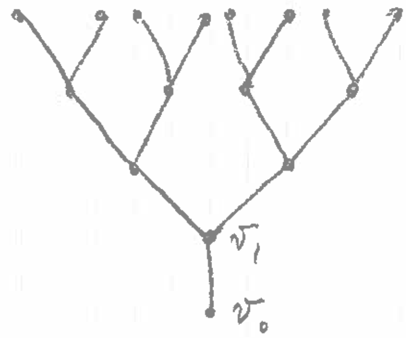
$$Z(\xi) = \frac{\xi}{1 - Z(\xi)} \Rightarrow Z(\xi) = \tfrac{1}{2}\left(1 - \sqrt{1 - 4\xi}\right)$$

$$\xi_0 = \tfrac{1}{4}, \quad Z_0 = \tfrac{1}{2} < \rho$$

$$Z_N = \frac{(2N-2)!}{N!\,(N-1)!} \sim \frac{1}{4\sqrt{\pi}}\, N^{-3/2}\, 4^N.$$

Ex. 3.2 (Percolation on a Cayley tree)

Let C be a fixed (infinite) Cayley tree with all vertices of order n except a root v_0 of order 1



Bondpercolation on C is described by the probability measure

$$P = \prod_{e \in E(C)} P_e \quad \text{on} \quad \prod_{e \in E(C)} \{0,1\}$$

where P_e is indep. of e and given by

$$P_e(1) = p \quad (e \text{ is open})$$
$$P_e(0) = 1-p \quad (e \text{ is closed})$$

except for the root edge $(x_0 x_1)$ which is by def. open. Here $0 \le p \le 1$.

Considering C as a family tree the probability that exactly m edges emerging from a vertex $x \neq x_0$ (outwards) are open is

$$p_m = \binom{n-1}{m} p^m (1-p)^{n-1-m}, \quad 0 \le m \le n-1.$$

Hence the probability of a given finite connected cluster τ of open edges containing $(v_0 v_1)$ is

$$P(\tau) = \prod_{v \in \tau \setminus r} p_{\sigma_v - 1}$$

and defines a Galton-Watson distribution on $\mathcal{T} \setminus \mathcal{T}_\infty$. Note that

$$P(\tau \text{ is finite}) = Z(1)$$

where $Z(\xi)$ is defined as above with $w_n = p_{n-1}$, $n \geq 1$, and

$$g(z) = \sum_{m=0}^{n-1} p_m z^m, \qquad \xi = \infty.$$

We say the G-W tree is subritical if $Z(1) = 1 \neq Z_0$, critical if $Z(1) = 1 = Z_0$ and supercritical if $Z(1) < 1$ (see fig. below).

Noting that

$$g(1) = 1, \quad g'(1) = \sum_{m=1}^{n-1} m p_m = (n-1)p$$

we see that bond percolation is critical if

$$p = p_c = \frac{1}{n-1},$$

subcritical if $p < p_c$ and supercritical if $p > p_c$.

<u>Incipient infinite cluster:</u> For $p = p_c$ P is a prob. measure on $\mathcal{T} \setminus \mathcal{T}_\infty$, i.e. clusters are finite with prob. 1 but average size diverges. Conditioning P on size N, the incipient infinite cluster is the limiting measure as $N \to \infty$. We construct this limit below.

taining $(v_0 v_1)$ is

$$P(\tau) = \prod_{v\in\tau\setminus v_0} p_{\sigma_v - 1}$$

and defines a Galton-Watson distribution on $T\setminus T_0$

We have

$$g(z) = \sum_{m=0}^{\infty} p_m z^m = \sum_{m=1}^{n-1} p_m z^m$$

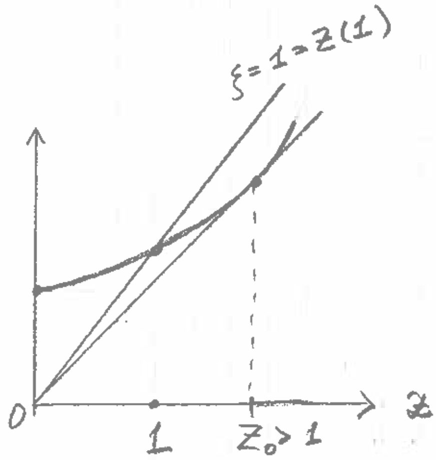
and

$$g(1) = 1, \quad g'(1) = \sum_{m=1}^{n-1} m p_m = (n-1)p$$

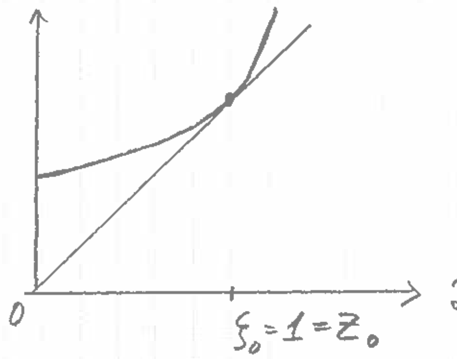
i.e. the G-W tree is critical if

$$p = p_c = \frac{1}{n-1}.$$

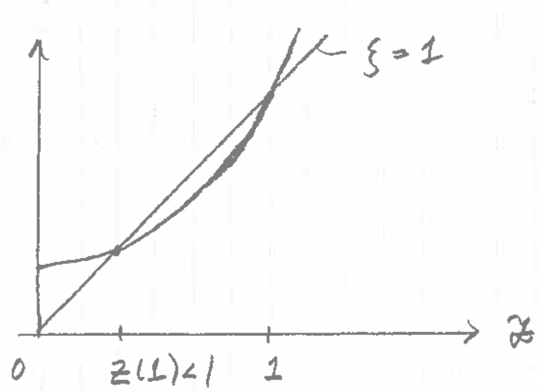
Subcritical $(p < p_c)$



Critical $(p = p_c)$



Supercritical $(p > p_c)$



3.2 Generic infinite random trees

Given weights (w_n) define probability distribution μ_N on $\mathcal{T}_N$ by

$$\mu_N(\tau) = \frac{w(\tau)}{Z_N}, \quad \tau \in \mathcal{T}_N.$$

Theorem 3.1 If the genericity assumption (II) holds for (w_n) the (weak) limit

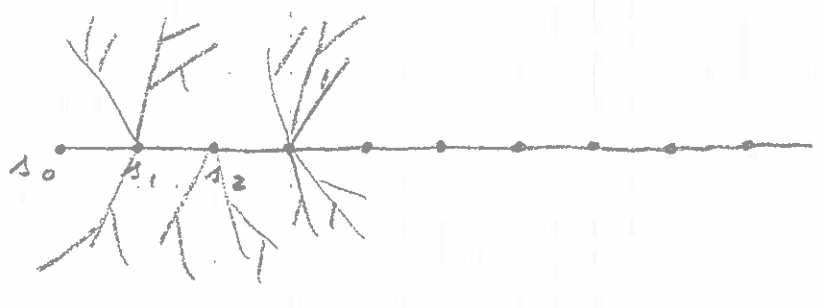
$$\mu = \lim_{N\to\infty} \mu_N$$

exists as a probability measure on $\mathcal{T}_\infty$, i.e.

$$\lim_{N\to\infty} \int f\, d\mu_N = \int f\, d\mu$$

for all bounded continuous fcts. f on $\mathcal{T}_\infty$.

Description of μ A tree $\tau \in \mathcal{T}_\infty$ has a single spine (or is one-ended) if there is a unique simple infinite path originating from v_0. This path is called the spine and is denoted by $(v_0 = s_0) s_1 s_2 s_3 \ldots$ Let $\mathcal{S}$ denote the subset of single spine trees in $\mathcal{T}_\infty$. Thus $\tau \in \mathcal{S}$ consists of the spine together with finite trees (branches) rooted at the spine vertices to the left and right (see fig. below).



Theorem 3.2 The measure μ is concentrated on $\mathcal{S}$ and is characterised as follows:

1) If $\mathcal{S}_n(k_1', k_1'', \ldots, k_n', k_n'')$ denotes the subset of $\mathcal{S}$ consisting of trees with k_i' left branches and k_i'' right branches rooted at s_i, $i=1,\ldots,n$, then

$$\mu(\mathcal{S}_n(k_1', k_1'', \ldots, k_n', k_n'')) = \prod_{i=1}^{n} (w_{k_i'+k_i''+2} Z_0^{k_i'+k_i''} \xi_0).$$

2) Conditioning on $\mathcal{S}_n(k_1', k_1'', \ldots, k_n', k_n'')$ each finite branch B at s_i is distributed independently of the others according to the grand canonical measure

$$P(B) = Z_0^{-1} \xi_0^{|B|} \prod_{v \in B \setminus s_i} w_{\sigma_v},$$

which is the critical G-W distribution corresponding to

$$p_n = \xi_0 w_{n+1} Z_0^{n-1}, \quad n \geq 0, \qquad (3.1)$$

Note. The last claim follows from:

i) $$\sum_{n=0}^{\infty} p_n = \xi_0 Z_0^{-1} g(Z_0) = 1 \quad \text{by (I)}$$

ii) $$\prod_{v \in B \setminus v_0} p_{\sigma_v - 1} = \xi_0^{|B|} \prod_{v \in B \setminus v_0} w_{\sigma_v} Z_0^{\sigma_v - 2}$$

$$= \xi_0^{|B|} Z_0^{-1} \prod_{v \in B \setminus v_0} w_{\sigma_v} = P(B)$$

where $\sum_{v \in B \setminus v_0} (\sigma_v - 2) = 2(B) - 1 - 2|B| = -1$ was used.

iii) $$\sum_{n=0}^{\infty} n p_n = \xi_0 \sum_{n=1}^{\infty} n w_{n+1} Z_0^{n-1} = \xi_0 g'(Z_0)$$
$$= \xi_0 Z_0^{-1} g(Z_0) = 1 \quad \text{by (III) and (I).}$$

In particular, for the <u>uniform tree</u> $w_n = 1, n \geq 1$, we get $\xi_0 = \frac{1}{4}$, $Z_0 = \frac{1}{2}$ and

$$p_n = \frac{1}{4} \cdot \left(\frac{1}{2}\right)^{n-1} = 2^{-(n+1)}, \quad n \geq 0.$$

<u>Sketch of proof of Thms 3.1 and 3.2</u>

Consider μ_N as a discrete measure on $\mathcal{T}$ considered as a complete separable metric space with metric d (see p. 12).

Let τ_0 be a finite tree of height R and define $\mathcal{C}(\tau_0) \subseteq \mathcal{T}$ by

$$\mathcal{C}(\tau_0) = \{\tau \in \mathcal{T} \mid B^{\tau}(R, v_0) = \tau_0\}.$$

<u>Claim</u>:

$$\lim_{N\to\infty} \mu_N(\mathcal{C}(\tau_0)) = M \cdot \xi_0^{|\tau_0| - M} Z_0^{M-1} \prod_{v \in B^{\tau_0}(R-1, v_0)} w_{\sigma_v}, \tag{3.2}$$

where $M = M(\tau_0)$ is the number of vertices in τ_0 at height R.

Proof: For $\tau \in \mathcal{C}(\tau_0)$ let $u_1, \dots, u_M$ denote the vertices at height R and let $N_1, \dots, N_M \geq$ denote the size of the subtrees $\tau_1, \dots, \tau_M$