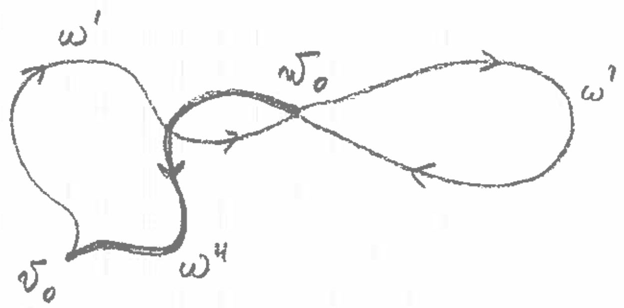
it leaves G_0. There are then $\sigma_{0,v}$ vertices in $V(G)$ it may hit next and each such walk contributes a probability $q_{\bar{G}}(\omega)$. This proves the claim. In case $\bar{G}_0$ is (connected and) recurrent this probability is 1. ∎

<u>Corollary 1.1</u> Assume G is connected and let $v_0, w_0 \in G$, $v_0 \neq w_0$. Then

$$\sum_{\substack{\omega: v_0 \to v_0 \\ w_0 \in \omega}} p_G(\omega) \leq \sigma_{v_0} \sigma_{w_0}^{-1} \sum_{\omega: v_0 \to w_0} p_G(\omega)$$

with equality holding if G is recurrent.

<u>Proof.</u> Every $\omega : v_0 \to v_0$ containing w_0 can be decomposed uniquely into a walk $\omega' : v_0 \to w_0$ and a walk $\omega'' : w_0 \to v_0$ such that ω'' does not return to w_0. Hence,



the reverse of ω'' is a walk ω''' in $G_0 = G \setminus w_0$ from v_0 to ∂G_0 and one additional step to w_0. Since $\sigma_{0,v} = 1$ for all $v \in \partial G_0$ in this case, Lemma 1 gives

$$\sum_{\substack{\omega : v_0 \to v_0 \\ w_0 \in \omega}} p_G(\omega) = \sum_{\omega' : v_0 \to w_0} q(\omega') \sum_{v \in \partial G_0} \sum_{\omega''' : v_0 \to v} q(\omega''') \sigma_{v_0}$$

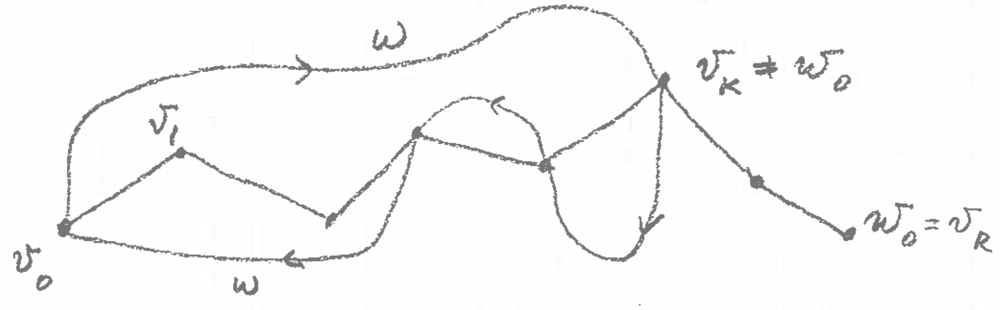
$$\leq \sum_{\omega' : v_0 \to w_0} \sigma_{v_0} \sigma_{w_0}^{-1} p_G(\omega') ,$$

with equality holding if $\overline{G}_0 = G$ is recurrent. $\square$

<u>Corollary 12</u> Assume G is connected and let $v_0, w_0 \in G$, $v_0 \neq w_0$. Then

$$\sum_{\substack{\omega : v_0 \to v_0 \\ w_0 \notin \omega}} p_G(\omega) \leq \sigma_{v_0} d_G(v_0, w_0) .$$

<u>Proof</u> Let $R = d_G(v_0, w_0)$ and choose a path $(v_0 v_1), (v_1 v_2), \ldots, (v_{R-1} w_0)$ from v_0 to $w_0 = v_R$. Given $\omega : v_0 \to v_0$ not containing w_0 there is a maximal $K = K(\omega) \leq R-1$ such that $v_K \in \omega$.



Then ω can be decomposed uniquely into a walk $\omega' : v_0 \to v_K$ and a walk $\omega'' : v_K \to v_0$ such that ω'' does not return to v_K or visit $v_{K+1}, \ldots, v_R$. From the previous proof it follows that

$$\sum_{\substack{\omega : v_0 \to v_0 \\ v_K \in \omega,\, v_{K+1}, \dots, v_{R-1}, v_0 \notin \omega}} \rho(\omega) \le \sigma_{v_0} \sigma_{v_K}^{-1} \sum_{\substack{\omega' : v_0 \to v_K \\ v_{K+1}, \dots, v_R \notin \omega'}} p_G(\omega')$$

$$\le \sigma_{v_0} \sum_{\substack{\omega' : v_0 \to v_K \\ v_{K+1} \notin \omega'}} q(\omega').$$

Using now Lemma 1 with $G_0 = G \setminus v_{K+1}$ we get that the last sum is ≤ 1 because $v_K \in \partial G_0$. Since this holds for each $K = 0, \dots, R-1$ the result follows. ∎

<u>Theorem 1.1</u> If G is a recurrent graph such that d_s and d_h exist then

$$d_s \ge \frac{2 d_h}{d_h + 1}.$$

<u>Proof.</u> Fix $v_0 \in G$ and note that

$$\sum_{v \in G} \sum_{\omega : v_0 \to v} p_G(\omega) \cdot (1-x)^{|\omega|/2} = (1 - \sqrt{1-x})^{-1} \le \frac{2}{x},$$

since

$$\sum_{v \in G} \sum_{\substack{\omega : v_0 \to v \\ |\omega| = n}} p_G(\omega) = 1$$

for each $n \ge 0$. Given $R > 0$ and $0 < x < 1$ it follows that there exists $v_{R,x} \in B^G(R, v_0)$ such that

$$\sum_{\omega : v_0 \to v_{R,x}} p_G(\omega)(1-x)^{|\omega|/2} \le \frac{2}{x\, |V(B^G(R, v_0))|}$$

Writing

$$Q_G(x) = \sum_{\substack{\omega: v_0 \to v_0 \\ v_{R,x} \in \omega}} p_G(\omega)(1-x)^{|\omega|/2}$$

$$+ \sum_{\substack{\omega: v_0 \to v_0 \\ v_{R,x} \notin \omega}} p_G(\omega)(1-x)^{|\omega|/2}$$

we note that the proof of Cor. 1 trivially generalizes to show that

$$\sum_{\substack{\omega: v_0 \to v_0 \\ w_0 \in \omega}} p_G(\omega)(1-x)^{|\omega|/2} \leq \sigma_{v_0} \sigma_{w_0}^{-1} \sum_{\omega: v_0 \to w_0} p_G(\omega)(1-x)^{|\omega|/2}$$

and hence we conclude from ( ) and Cor. 2 that

$$Q_G(x) \leq \frac{2\sigma_{v_0}}{x\,|V(B^G(R,v_0))|} + \sigma_{v_0} \cdot R$$

$$\leq \frac{C}{x \cdot R^{d_h}} + \sigma_{v_0} \cdot R$$

for some constant $C > 0$ and R large enough. Now choose $R = R(x) = [x^{-(d_h+1)^{-1}}]$ and get

$$Q_G(x) \leq C' x^{-(d_h+1)^{-1}}$$

and hence

$$\alpha \leq (d_h+1)^{-1}$$

i.e.

$$d_s \geq 2(1-(d_h+1)^{-1}) = \frac{2d_h}{d_h+1}. \quad \square$$

2. Random graphs

Def. A random graph is an ensemble $\mathcal{G}$ of (connected) graphs equipped with a probability measure μ. Assume graphs are rooted with root v_0 and $\mathcal{G}$ is equipped with the metric

$$d(G_1, G_2) = \inf\left\{ \frac{1}{R+1} \;\middle|\; B^{G_1}(R, v_0) = B^{G_2}(R, v_0) \right\}.$$

Drop v_0 from notation.

Annealed Hausdorff dimension:

$$\bar{d}_h = \lim_{R\to\infty} \frac{\ln \langle |V(B^G(R))| \rangle_\mu}{\ln R}.$$

Annealed spectral dimension:

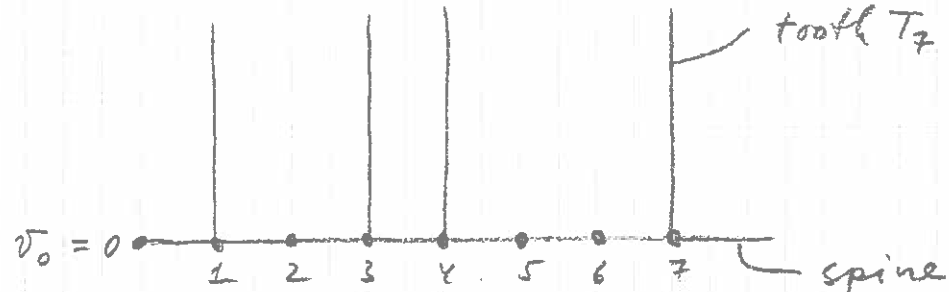
$$\bar{d}_s = 2(1-\alpha)$$

if

$$\langle Q_G(x) \rangle_\mu \sim x^{-\alpha} \quad \text{as} \quad x \to 0,$$

where $0 \le \alpha \le 1$.

EX. 2.1 (Bernoulli comb) Let $\mathcal{G} = \mathcal{C}$ consist of combs whose teeth have length 0 or ∞:



Let μ be the probability distribution defined by

$$\mu(|T_i| = \infty) = p, \quad i = 1, 2, 3, \ldots$$

where $0 \le p \le 1$ is fixed, and $\{|T_i| = \infty\}$ and $\{|T_j| = \infty\}$ are indep. events for $i \neq j$.

Theorem 2.1 For the Bernoulli comb with $p > 0$ there exist finite constants $\Lambda > 0$ and $\Lambda_C > 0$ for almost every $C \in \mathcal{G}$ such that

$$x^{-1/4} \le Q_C(x) \le \Lambda_C \, x^{-1/4} |\log x|^{1/2}$$

$$x^{-1/4} \le \langle Q_C(x) \rangle_\mu \le \Lambda \, x^{-1/4} |\log x|^{1/2}$$

for $0 < x < \frac{1}{2}$. In particular, $\bar{d}_s = \frac{3}{2}$ and $d_s = \frac{3}{2}$ almost surely.

Sketch of proof Repeated use of Lemma 1.2 yields the following

Monotonicity lemma For a comb C we have that $P_C(x)$ is a decreasing function of the length, ℓ_k, of the tooth T_k for any $k \ge 1$.

Rearrangement lemma Let C' be the comb obtained from C by swapping the teeth T_k and T_{k+1}. Then $P_C(x) > P_{C'}(x)$ if and only if $\ell_k < \ell_{k+1}$.

The lower bounds follow immediately from the Monotonicity lemma and Ex. 1.3

Upper bounds: Let L_k denote the distance between the $(k-1)$'th and k'th infinite tooth. Then L_k, $k \geq 1$, are independent random variables and

$$\mu(\{L_1, \dots, L_N \leq L\}) = (1-q^L)^N$$

where $q = 1-p < 1$. Choosing $N(x) = [x^{-\frac{1}{2}-\varepsilon}]$ and $L(x) = [x^{-\varepsilon}]$, where $\varepsilon > 0$ is small it follows that, if

$$\mathcal{A}_\varepsilon = \{L_i > L \text{ for some } i = 0, \dots, N(x)\},$$

then

$$\mu(\mathcal{A}_\varepsilon) \leq e^{-cx^{-\varepsilon}}$$

for some $c > 0$.

For $C \notin \mathcal{A}_\varepsilon$ one removes all teeth T_k, $k \geq N(x)$, and rearranges the remaining teeth to have constant spacing L. By the Monotonicity and Rearrangement lemmas one has

$$P_C(x) \leq P_L^{(1)}(x) + P_L^{(2)}(x)$$

where $P_L(x)$ is the gen. fct for first return probability for the rearranged comb C_L and $P_L^{(1)}(x)$ is the contribution from walks not hitting the point $N(x)$ on the spine and $P^{(2)}(x)$ the remainder. It is easy to see that