Recurrence properties of random infinite graphs.

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0. Introduction

- Not Renyi-Erdős random graphs
- Local weights, infinite size limits:

a) Two-dim quantum gravity (ADJ: Quantum Geometry 1997)

b) Percolation (G.R. Grimmett: Percolation 1999)

c) Statistical systems in random environments, D. ben-Avraham, S. Havlin: Diffusion and reactions in Fractals and Disordered Systems 2000

1. Recurrence, spectral dim. and Hausdorff dim.

Graph $G = (V, E)$, V = vertex set, E = edge set unoriented, countable, locally finite

Size: $|G| = |E(G)|$

Degree of $v \in V(G) = \#(\text{edges containing } v) = \sigma_v$

Path: sequence of different edges $(v_0 v_1), (v_1 v_2), \ldots, (v_{k-1} v_k)$

Connected graph: Any pair of vertices are connected by a path. Assumed in the following.

Graph distance:

$$d_G(v, v') = \text{minimal size of path connecting } v \text{ and } v'$$

Planar graph

Walk: $\omega : v \to v'$ is a sequence of edges

$$(v_0 v_1), (v_1 v_2), \ldots, (v_{k-1} v_k), \quad v_0 = v, \quad v_k = v'$$

Simple random walk on G:

$$p_G(\omega) = \prod_{i=0}^{|\omega|-1} \sigma_{\omega(i)}^{-1},$$

where $\omega(i)$ is the i'th vertex on ω.

Defines a probability distr. p_G^n on walks ω of length n originating at some fixed v_0.

Also defines a probability distribution p_G^∞ on infinite walks originating at some fixed v_0 by

$$p_G^\infty(\{\omega : v_0 \to * \mid \omega(i) = \omega_0(i),\ i = 0, \dots, n\})$$

$$= p_G^n(\omega_0)$$

where $\omega : v_0 \to *$ denotes an arb. infinite walk originating at v_0 and ω_0 is any given walk of length n originating at v_0.

<u>Def. 1.1</u> G is recurrent if, given v_0, we have

$$p_G^\infty(\{\omega : v_0 \to * : \exists i > 0 : \omega(i) = v_0\}) = 1,$$

or

$$\sum_{n=1}^{\infty} p_G^\infty(A_n) = 1$$

where

$$A_n = \{\omega : v_0 \to * \mid \omega(n) = v_0,\ \omega(i) \neq v_0 \text{ for } 1 \leq i \leq n-1\}.$$

$$p_G^\infty(A_n) = p_G^w(\{\omega : v_0 \to v_0 \mid \omega(i) \neq v_0 \text{ for } 1 \leq i \leq n-1\})$$

<u>Lemma 1.1</u> Let

$$B_n = \{\omega : v_0 \to * \mid \omega(n) = v_0\}.$$

Then G is recurrent if and only if

$$\sum_{n=1}^{\infty} p_G^\infty(B_n) = \infty.$$

<u>Proof.</u> Let i_k denote the time of k'th return to v_0, $k = 0,1,2,\dots$ Then

$$p_G^\infty(B_n) = \sum_{l=0}^{n} p_G^\infty(i_l = n)$$

$$= \sum_{l=0}^{n} \sum_{1\le n_1<n_2<\dots<n_l=n} p_G^\infty(i_1=n_1,\dots,i_{l-1}=n_{l-1}, i_l=n)$$

$$= \sum_{l=0}^{\infty} \sum_{1\le n_1<\dots<n_l=n} \prod_{i=1}^{l} p_G^\infty(A_{n_i-n_{i-1}})$$

where $n_0 = 0$. Summing over n gives

$$\sum_{n=0}^{\infty} p_G^\infty(B_n) = \sum_{l=0}^{\infty} \left(\sum_{m=1}^{\infty} p_G^\infty(A_m)\right)^l$$

which proves the claim. $\square$

<u>Example 1.1</u> a) Any finite graph is recurrent: It is easy to see that $\sum_{n=1}^{\infty} p_G^\infty(B_n) = \infty$.

b) $\mathbb{Z}^d$ is recurrent if $d = 1,2$ and not recurrent (trancient) if $d \ge 3$:

Use Fourier transformation: Set

$$G(k) = \sum_{v\in\mathbb{Z}^d} \sum_{w: v_0\to v} p_G(w)\, e^{-ik(v-v_0)}$$

$$= \sum_{v\in\mathbb{Z}^d} \sum_{w: v_0\to v} \prod_{i=0}^{|w|-1} (2d)^{-1} e^{-ik(w(i+1)-w(i))}$$

$$= \sum_{n=0}^{\infty} \left((2d)^{-1} \sum_{\alpha=1}^{d} \left(e^{-ik_\alpha} + e^{+ik_\alpha}\right)\right)^n$$

$$= \left(1 - d^{-1} \sum_{\alpha=1}^{d} \cos k_\alpha \right)^{-1}, \quad -\pi \le k_\alpha \le \pi .$$

Integrating $G(k)$ w.r.t. k over $[-\pi, \pi]^d$ it is seen that only $v = v_0$ contributes with the result

$$(2\pi)^d \sum_{\omega : v_0 \to v_0} p_G(\omega) = (2\pi)^d \sum_{n=1}^{\infty} p_G^{\infty}(B_n) .$$

On the other hand it is seen that $G(k)$ is integrable if and only if $d \geq 3$. ∎

<u>Def. 1.2</u> (Generating fcts. for return probabilities)

$$P_G(x) = \sum_{n=1}^{\infty} p_G^{\infty}(A_n)(1-x)^{n/2}$$

$$Q_G(x) = \sum_{n=0}^{\infty} p_G^{\infty}(B_n)(1-x)^{n/2}$$

Note

$$Q_G(x) = \frac{1}{1 - P_G(x)}$$

and G is recurrent if and only if $Q_G(x)$ is divergent at $x = 0$:

$$Q_G(x) \sim x^{-\alpha} \quad \text{as} \quad x \to 0$$

$$1 - P_G(x) \sim x^{\alpha} \quad \text{as} \quad x \to 0 .$$

<u>Spectral dimension</u> of G:

$$d_s = 2(1-\alpha) \qquad (\text{if } \alpha > 0)$$

$$\alpha = 0 : \quad Q_G(x) \sim |\log x|^{\beta}, \quad \beta > 0, \quad d_s = 2 .$$

Ex. 1.2 $G = \mathbb{Z}_+$. $\qquad P_G(x) = P_\infty(x)$

$$P_\infty(x) = \frac{\frac{1}{2}(1-x)}{1 - \frac{1}{2}P_\infty(x)}$$

$$\Rightarrow \quad P_\infty(x)^2 - 2P_\infty(x) + (1-x) = 0$$

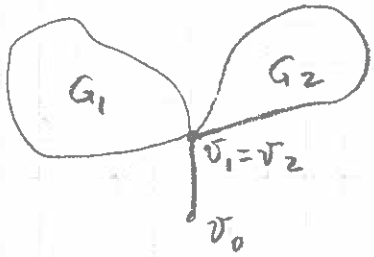
$$\Rightarrow \quad P_\infty(x) = 1 - \sqrt{x}$$

i.e. $\quad \alpha = \frac{1}{2}, \quad d_s = 1.$

Lemma 1.2 Let G_1 and G_2 be two connected disjoint graphs, $v_1 \in V(G_1)$, $v_2 \in V(G_2)$, and let G_0 be obtained by identifying v_1 and v_2 and adding a new vertex v_0 as well as the link $(v_0 v_1) = (v_0 v_2)$. Then

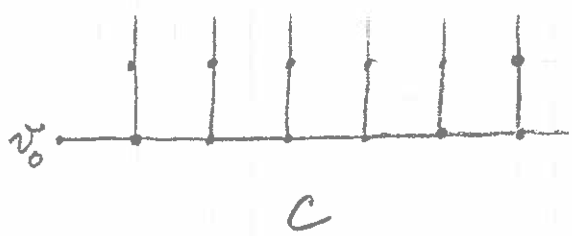
$$P_{G_0}(x) = \frac{1-x}{1 + \sigma_{v_1} + \sigma_{v_2} - P_{G_1}(x) - P'_{G_2}(x)}$$

Proof.



$\square$

Ex. 1.3 (Comb with infinite teeth.)



Here $G_1 = \mathbb{Z}_+$ and $G_2 = C$, $\sigma_{v_1} = \sigma_{v_2} = 1$.

$$P_C(x) = \frac{1-x}{3-P_\infty(x)-P_C(x)} = \frac{1-x}{2+\sqrt{x}-P_C(x)}$$

$$\Rightarrow \quad P_C(x) = 1 - x^{1/4}\sqrt{1+\tfrac{5}{4}\sqrt{x}} + \tfrac{1}{2}\sqrt{x}.$$

i.e. $\alpha = \frac{1}{4}$ and $d_s = \frac{3}{2}$.

<u>Hausdorff dimension</u> G connected, $v_0 \in V(G)$. Let $B^G(R, v_0)$ be the ball of radius R centered at v_0, i.e. $B^G(R, v_0)$ is the subgraph of G spanned by vertices at graph distance $\leq R$ from v_0. Then

$$d_h = \lim_{R\to\infty} \frac{\log |V(B^G(R, v_0))|}{\log R}$$

if it exists.

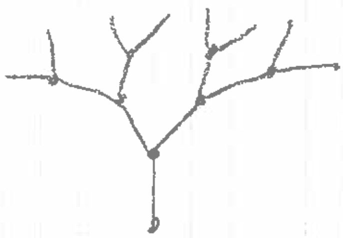
<u>Ex. 1.4</u> a) $d_h = 0$ if G is finite.

b) $d_h = d$ if $G = \mathbb{Z}^d$.

c) $d_h = 1$ if $G = \mathbb{Z}_+$

d) $d_h = 2$ if $G = C$. Note: $d_s \neq d_h$.

e) $d_h = \infty$ if G is a Cayley graph:



Note: in this case Lemma 2 gives

$$P_G(x) = \frac{1-x}{3-2P_G(x)}$$

$$\Rightarrow \quad 2P_G(x)^2 - 3P_G(x) + 1 - x = 0$$

$$\Rightarrow \quad P_G(x) = \frac{3-\sqrt{9-8(1-x)}}{4} = \frac{3}{4} - \frac{1}{4}\sqrt{1+8x}$$

Hence $P_G(0) = \frac{1}{2} < 1$. G not recurrent.

Relation between d_s and d_h.

Let G be a graph and G_0 a subgraph of G. By ∂G_0 we denote the subgraph spanned by the vertices of G_0 having at least one neighbour in $V(G)\setminus V(G_0)$ and by $\overline{G}_0$ we denote the subgraph spanned by G_0 and nearest neighbours of vertices in G_0. For $v \in V(G_0)$ we define the out-degree $\sigma_{0,v}$ of v as the number of neighbours of v in $V(G)\setminus V(G_0)$.

<u>Lemma 1.3</u> Let G be a connected graph and G_0 a subgraph such that $\partial G_0 \neq \phi$. Setting

$$q_G(\omega) = \prod_{i=0}^{|\omega|} \sigma_{\omega(i)}^{-1}$$

for a walk in G we have, for arbitrary $v_0 \in G_0$, that

$$\sum_{v\in\partial G_0} \sum_{\substack{\omega: v_0\to v \\ \omega\in G_0}} q_G(\omega)\, \sigma_{0,v} \leq 1 ,$$

with equality holding if $\overline{G}_0$ is connected and recurrent.

<u>Proof</u> We claim that the LHS is the probability that a walk ω starting at v_0 leaves G_0: Given such an ω let $v \in \partial G_0$ denote the last vertex visited in G_0 before