Part II. The Ising model

G finite connected graph, vertex set V, edge set E

$J = (J_e)_{e \in E} \in (0,\infty)^E$ positive edge-weights

A <u>spin configuration</u> on G is an element $\sigma \in \Omega(G) = \{\pm 1\}^V$

The <u>energy</u> of $\sigma \in \Omega(G)$ is $\mathcal{H}(\sigma) := -\sum_{e=\{u,v\}\in E} J_e \sigma_u \sigma_v$ (lower if nbging spins are aligned)

This induces a probability measure $\mu_{G,\beta}$ on $\Omega(G)$: $\mu_{G\beta}(\sigma) := \frac{1}{Z^J_\beta(G)} e^{-\beta \mathcal{H}(\sigma)}$ (higher if spin ali

where $Z^J_\beta(G) := \sum_{\sigma \in \Omega(G)} e^{-\beta \mathcal{H}(\sigma)}$ is the <u>partition function</u> for the Ising model on (G,J),

and $\beta = \frac{1}{T} \geq 0$ is the <u>inverse temperature</u>.

<u>Questions</u>: 1. Can we compute efficiently the partition function ?

2. How does the system evolve when T varies ?

We shall first make the statement of 2. more precise (phase transition), then give a famous heuristic partial answer to

than answer 1, then answer 2 for bipercolative graphs.

II. 1. Phase transition

So, how does the prob. measure $\mu_{G\beta}$ depend on T ? Let's begin with extreme cases :

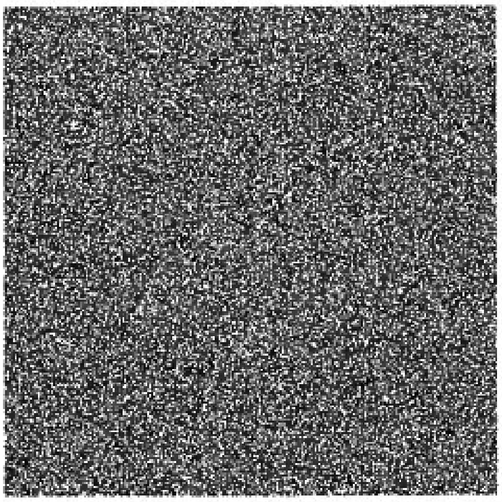
- $T \to \infty \iff \beta = 0 \Rightarrow$ all σ's are equiprobable (σ_v are ind. Bernoulli variables).
- $T \to 0 \iff \beta \to \infty \Rightarrow$ only 2 σ's have pos. prob : $\sigma \equiv +1$ and $\sigma \equiv -1$.

For G finite, everything varies smoothly in $\beta = \frac{1}{T} \in (0,\infty)$.

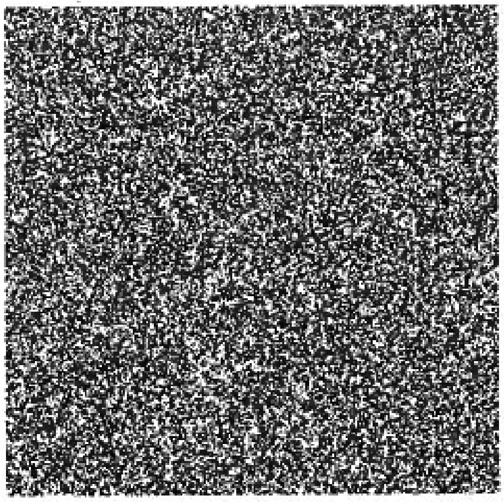
But now, imagine G_n finite $\to \mathcal{G}$ infinite graph.

Example : $G_n =$ $n\times n$ square lattice $\subset \mathbb{Z}^2$ $J \equiv 1$ (slide : $n = 500$, $p_\beta = 1-e^{-2\beta} \in [0,1]$: symmetry break at $p_\beta \approx 0.59$ ($\beta \approx 0.44..$)

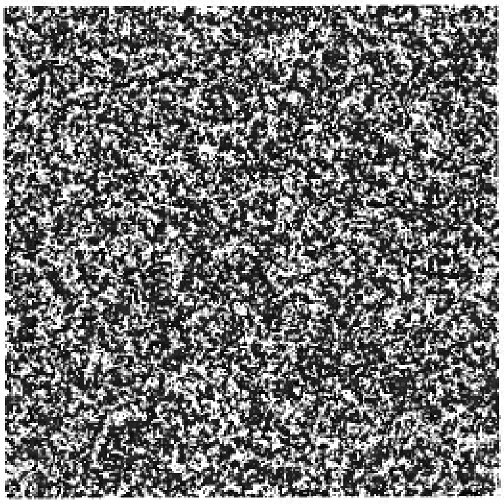
(Simulations by Yvan Velenik)



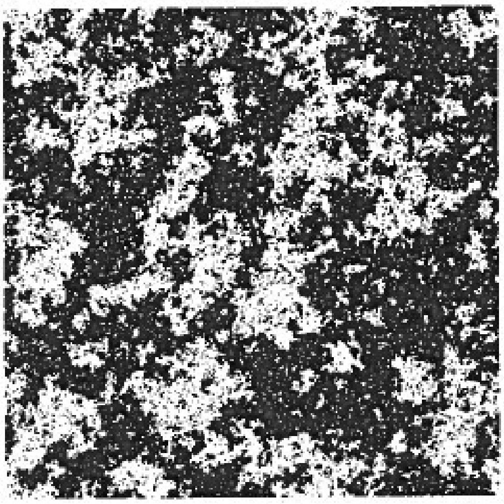
$p_\beta = 0$



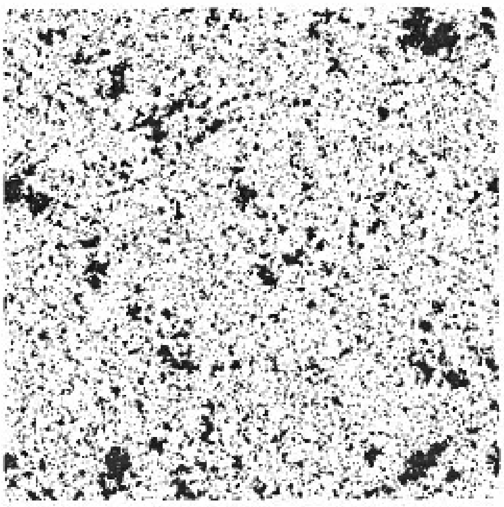
$p_\beta = 0{,}4$



$p_\beta = 0{,}5$



$p_\beta = 0{,}58$



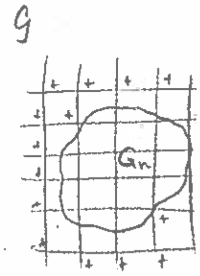
$p_\beta = 0{,}59$

$p_\beta = 0{,}7$

Configurations typiques du modèle d'Ising en dimension 2 avec condition au bord périodique ($N = 500$), pour différentes valeurs du paramètre $p_\beta \stackrel{\text{déf}}{=} 1 - e^{-2\beta}$.

"Experimental observation" of a phase transition : a sharp qualitative change in the model at some critical $\beta = \beta_c \in (0,\infty)$

Here is one possible rigorous definition of β_c (among many) :



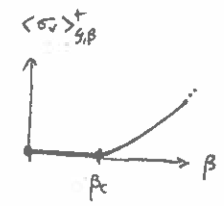
let G_n finite $\nearrow$ $\mathcal{G}$ infinite,

let $\mu^+_{G_n,\beta}$ denote the Ising prob. measure on $G_n \subset \mathcal{G}$ conditioned with $+$-boundary condition :

Standard fact (follows from FKG-inequalities) : the measures $\mu^+_{G_n,\beta} \xrightarrow{n\to\infty} \mu^+_{\mathcal{G},\beta}$ ($:\Leftrightarrow$ $\forall f$ local fct (i.e. det. by a finite nb of σ_v's), $\langle f\rangle^+_{G_n,\beta} \to \langle f\rangle^+_{\mathcal{G},\beta}$) indep. of the choice of $G_n \nearrow \mathcal{G}$.

In particular, for any fixed $v \in V(\mathcal{G})$, we have an expectation $\langle \sigma_v \rangle^+_{\mathcal{G},\beta} = \lim_{n\to\infty} \langle \sigma_v\rangle^+_{G_n,\beta}$.

Fact (GKS-inequalities) : $\beta \longmapsto \langle \sigma_v \rangle^+_{\mathcal{G},\beta}$ is mon. increasing



Clearly, $\langle \sigma_v \rangle^+_{\mathcal{G},0} = 0$ $\Rightarrow$ define $\beta_c := \sup\{\beta \geq 0 \mid \langle \sigma_v\rangle^+_{\mathcal{G},\beta} = 0\} \in [0,\infty]$

(only depends on $(\mathcal{G},J)$, not on $G_n \nearrow \mathcal{G}$, nor on v)

Examples : • $\mathcal{G}$ finite $\Rightarrow$ $\beta_c = 0$, no phase transition

• $\mathcal{G}$: $\cdots$—•—•—•$\cdots$ $\Rightarrow$ $\beta_c = \infty$ (Ising's PhD, 1920's), no phase transition.

Fact • (Peierls, 1936) : For a huge class of infinite weighted graphs $(\mathcal{G},J)$, including all planar biperiodic graphs (with faces top. discs), $0 < \beta_c < \infty$.

$\rightsquigarrow$ Question. How can one determine β_c ?

We shall start with a heuristic argument for #.

II.2. Kramers-Wannier duality

Ⓐ. High-temperature representation ('van der Waerden, 1941)

Let (G, J) be an arbitrary finite abstract weighted graph.

$$\exp(\beta J_e \sigma_u \sigma_v) = \cosh(\beta J_e)\big(1 + \overbrace{\tanh(\beta J_e)}^{=: x_e}\sigma_u\sigma_v\big)$$

$$Z_\beta^J(G) = \sum_\sigma e^{-\beta \mathcal{H}(\sigma)} = \sum_\sigma \prod_{e=\{u,v\}\in E} \exp(\beta J_e \sigma_u \sigma_v) = \underbrace{\prod_{e\in E}\cosh(\beta J_e)}_{=C} \sum_\sigma \underbrace{\prod_{e=\{u,v\}\in E}(1 + x_e\sigma_u\sigma_v)}_{\sum_{\gamma\subset E} \underbrace{\prod_{e\in\gamma} x_e}_{x(\gamma)} \prod_{v\in V}\sigma_v^{\deg_\gamma(v)}}$$

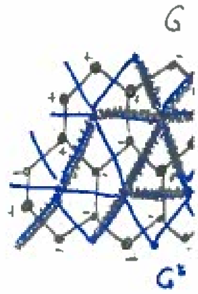
$$= C\cdot\sum_{\gamma\subset E} x(\gamma) \underbrace{\sum_{\sigma: V\to\{\pm 1\}} \prod_{v\in V}\sigma_v^{\deg_\gamma(v)}}_{\prod_{v\in V}(1 + (-1)^{\deg_\gamma(v)})} = C\cdot 2^{|V|}\sum_{\gamma\in\mathcal{E}(G)} x(\gamma),$$

where $\mathcal{E}(G) = \{\gamma \subset G \mid \forall v\in V(G),\ \deg_\gamma(v) \text{ is even}\} = \mathrm{Ker}(C_1 \to C_0)$, cycles modulo 2.

$$\Rightarrow \boxed{Z_\beta^J(G) = C\cdot\sum_{\gamma\in\mathcal{E}(G)} x(\gamma), \quad \text{where } x(\gamma) = \prod_{e\in\gamma} x_e, \quad x_e = \tanh(\beta J_e)}$$

($\forall G$ abstract finite graph)

Ⓑ Low-temperature representation



Now, assume (G, J) is a planar weighted graph, and let G^* be the dual graph.

To $\sigma\in\Omega(G)$, one can associate $\gamma(\sigma) := \{e^*\in E(G^*) \mid \ \}\in\mathcal{E}(G^*)$.

This map $\Omega(G)\to\mathcal{E}(G^*)$ is clearly 2 to 1: $\gamma(\sigma) = \gamma(\bar\sigma)$.

By definition, its image is the set of boundaries of G^* $= \mathrm{Im}(C_2\to C_1) \overset{\text{planar!}}{=} \mathrm{Ker}(C_1\to C_0) = \mathcal{E}(G^*)$.

So we have a surjective 2 to 1 map $\Omega(G)\to\mathcal{E}(G^*)$.

Furthermore, $$\mathcal{H}(\sigma) = -\sum_{e=\{u,v\}\in E(G)} J_e\sigma_u\sigma_v = \sum_{e^*\in\gamma(\sigma)} J_e - \sum_{\substack{e=\{u,v\}\in E\\ \sigma_u=\sigma_v}} J_e = 2\sum_{e^*\in\gamma(\sigma)} J_e - \sum_{e\in E} J_e$$

$$\Rightarrow e^{-\beta\mathcal{H}(\sigma)} = e^{\beta\sum_e J_e}\cdot\prod_{e^*\in\gamma(\sigma)}\exp(-2\beta J_e)$$

$$\Rightarrow \boxed{Z_\beta^J(G) = C'\cdot\sum_{\gamma^*\in\mathcal{E}(G^*)} x^*(\gamma), \quad \text{where } x^*(\gamma) = \prod_{e\in\gamma} x_e^*, \quad x_e^* = \exp(-2\beta J_e)}$$

x small $\Leftrightarrow$ x^* big

$\downarrow$

(c) Duality

Note that the weights $x_e = \tanh(\beta J_e) \in [0,1]$ and $x_e^* = \exp(-2\beta J_e) \in [0,1]$ are related by $x + x^* + x x^* = 1$ (dual weights). Equating both equations above and using at the constant, one gets:

$$\boxed{2^{|V|/2} \prod_e (1+x_e)^{-1/2} \sum_{\gamma \in \mathcal{E}(G)} x(\gamma) = 2^{|V^*|/2} \prod_e (1+x_e^*)^{-1/2} \sum_{\gamma \in \mathcal{E}(G^*)} x^*(\gamma)}$$

Furthermore, setting $J \equiv 1$ ($\Rightarrow x = \tanh(\beta)$, $x^* = e^{-2\beta}$) and choosing β^* s.t. $\tanh(\beta^*) = e^{-2\beta}$, we get:

$$f_G(\beta^*) := \lim_{n\to\infty} \frac{1}{|V(G_n)|} \log Z_{\beta^*}^{1}(G_n) \overset{\text{high}}{=} (\text{analytic fct of } \beta) + \lim_{n\to\infty} \frac{1}{|V(G_n)|} \log \Big(\sum_{\gamma \in \mathcal{E}(G)} \overbrace{x(\gamma)}^{= x^*(\gamma)} \Big)$$

$$\overset{\text{low}}{=} (\text{analytic fct of } \beta) + f_{G^*}(\beta)$$

This relates the free energy of G to the free energy of G^*.

Now, the relation $\tanh(\beta^*) = e^{-2\beta}$ sends β small to β^* big and vice versa,

fixing $\beta_{sd} = \frac{1}{2}\log(1+\sqrt{2}) \approx 0.441\ldots$

Consider the case $G =$ , $J \equiv 1$. Then, $G^* = G \Rightarrow f_G(\beta^*) = (\text{analytic}) + f_G(\beta)$.

Since f_G is non-analytic at β_c (fact), assuming this is the only singularity, we must have $\beta_c = \beta_{sd}$.

(Kramers–Wannier, 1941).

Rigorous proof: Onsager, 1944.

In the rest of the class, we will:

- show of to compute efficiently the high temp. expansion for any finite graph
- extend the K-W duality displayed above to non-planar graphs
- determine β_c for any biperiodic weighted planar graph
- show that f_G is analytic at $\beta \neq \beta_c$ for any such graphs.