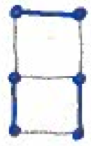
Part I. The dimer model

Γ finite connected graph, $V(\Gamma)$ vertices, $E(\Gamma)$ edges

A <u>dimer configuration</u> on Γ (aka <u>perfect matching</u>) is a choice of edges of Γ, called <u>dimers</u>, s.t. each vertex of Γ is adjacent to exactly one dimer.

Set $\mathcal{D}(\Gamma) := \{\text{dimer conf. on } \Gamma\}$.

Ex: 

Rem: $\mathcal{D}(\Gamma) \neq \emptyset \Rightarrow \# V(\Gamma)$ even, ($\not\subset$, $\curlywedge$).

So we'll always assume $\# V(\Gamma)$ even

Let $\nu : E(\Gamma) \to (0,\infty)$ be an edge-weight system. This induces a prob. measure on $\mathcal{D}(\Gamma)$.

$$\mathrm{Prob}(D) = \frac{\nu(D)}{Z}, \quad \text{where } \nu(D) = \prod_{e\in D} \nu(e) \quad \text{and} \quad Z = Z_{\text{dimer}}(\Gamma,\nu) = \sum_{D\in\mathcal{D}(\Gamma)} \nu(D).$$

The study of this measure is called the <u>dimer model</u> on Γ, and Z is its <u>partition function</u>.

Rem: if $\nu \equiv 1$, then $Z = \# \mathcal{D}(\Gamma)$.

It turns out that one can prove a lot of things about this model, and a lot of this rests on one basic fact: "given Γ and ν, one can often compute Z efficiently."

The goal of today's lecture is to explain this statement.

I.1. Dimers and Pfaffians

$A = (a_{ij})_{i,j=1}^{2n}$ skew-symmetric $\Rightarrow \det(A)$ is the square of the <u>Pfaffian</u> of A,

$$\mathrm{Pf}(A) := \frac{1}{2^n n!} \sum_{\sigma\in S_{2n}} \underbrace{\overset{\text{sign of } \sigma}{(-1)^{\sigma}}\, a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(2n-1)\sigma(2n)}}_{a^{\sigma}}$$

Rem: • Like the determinant, Pf can be computed efficiently ($O(n^3)$ operations).

• $\mathrm{Pf}\begin{pmatrix} \overset{n}{0} & \overset{n}{M} \\ -M^T & 0 \end{pmatrix}\!\begin{matrix} \}n \\ \}n \end{matrix} = (-1)^{\frac{n(n-1)}{2}} \det(M)$

$\sigma \in S_{2n}$ induces a partition P of $\{1,2,..,2n\}$ into unordered pairs $\{\sigma(1),\sigma(2)\}, \dots, \{\sigma(2n-1),\sigma(2n)\}$,

and σ, σ' give same partition $\Rightarrow$ $a^{\sigma} = a^{\sigma'}$ $\quad$ ($a_{ij} = -a_{ji}$)

There are $2^n \cdot n!$ such σ's for each P, so: $\quad \mathrm{Pf}(A) = \sum_{P \text{ partition}} (-1)^{\sigma} a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(2n-1)\sigma(2n)}$

<u>Idea</u> [Fisher, Kasteleyn, Temperley, 1961] :

Given (Γ, ν), number the vertices $1, 2, .., 2n$.

Fix an arbitrary orientation K of the edges of Γ, and let $A^K = A^K(\Gamma,\nu)$

be the associated weighted skew-adjacency matrix, i.e.:

$$A^K = (a^K_{ij})_{i,j=1}^{2n} \quad , \quad a^K_{ij} := \begin{cases} \nu(e) & \text{if } i \xrightarrow{e} j \\ -\nu(e) & \text{if } i \xleftarrow{e} j \\ 0 & \text{if } i \not\sim j . \end{cases} =: \varepsilon^K_{ij} \cdot \nu(e).$$

Then:

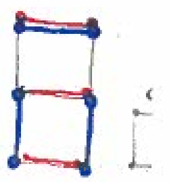
$$\mathrm{Pf}(A^K) = \sum_{P=[\sigma] \in \text{part}} (-1)^{\sigma} a^K_{\sigma(1)\sigma(2)} \cdots a^K_{\sigma(2n-1)\sigma(2n)} = \sum_{[\sigma]=D \in \mathcal{D}(\Gamma)} \overbrace{(-1)^{\sigma} \varepsilon^K_{\sigma(1)\sigma(2)} \cdots \varepsilon^K_{\sigma(2n-1)\sigma(2n)}}^{\varepsilon^K(D)} \cdot \nu(D) = \sum_{D \in \mathcal{D}(\Gamma)} \varepsilon^K(D) \nu$$

$\Rightarrow$ can we find an orientation K st $\varepsilon^K(D) = \varepsilon^K(D')$ $\forall D, D' \in \mathcal{D}(\Gamma)$? $\quad$ ($\Rightarrow Z = |\mathrm{Pf}(A^K)|$)

To solve this, let's consider $D, D' \in \mathcal{D}(\Gamma)$, and study $\varepsilon^K(D)\,\varepsilon^K(D')$.

Note that $D, D' \in \mathcal{D}(\Gamma)$, $\quad D \Delta D' := (D \cup D') \setminus (D \cap D') = \bigsqcup_i C_i$, C_i cycle of even length. (<u>composition cycle</u>

<u>Easy fact</u>: $\quad \varepsilon^K(D) \cdot \varepsilon^K(D') = \prod_i (-1)^{n^K(C_i)+1}$,



where $n^K(C) = \#\{ \text{edges in } C \mid \xrightarrow{C} \neq K \}$ $\quad$ (parity ind. of choice of orient of C)

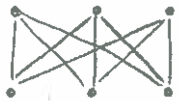
$\Rightarrow$ we need to find an orientat. K s.t. $n^K(C)$ is odd $\forall C$ comp. cycle.

Definition: An orientation K on Γ is <u>pfaffian</u> if, for any cycle C of even length s.t. $\Gamma \setminus V(C)$ admits a dimer conf., $n^K(C)$ is odd. A graph is <u>pfaffian</u> if it admits such an orientation.

(K pfaff $\Rightarrow$ $Z = |\mathrm{Pf}(A^K)|$)

Rem: If Γ is <u>bipartite</u>, i.e. $V(\Gamma) = B \sqcup W$ with no edge btw vert of same color, then $A^K = \begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix}$

So, if $\#B = \#W$ and K is pfaffian, $\quad Z = |\det(M)|$.

Bad news : Some graphs are not pfaffian, eg.  $=: K_{3,3}$

Good news : It works for planar graphs.

I.2. Kasteleyn's theorem

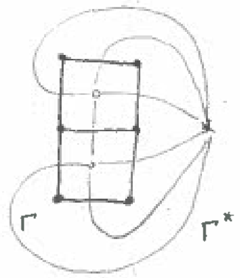
> **Theorem [Kasteleyn, 1961] :**
>
> Every planar graph is pfaffian

Let's follow his proof. Fix $\Gamma \subset \mathbb{R}^2$

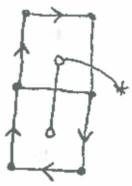
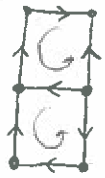
oriented as boundary of face $f \subset \mathbb{R}^2$, i.e. $\circlearrowleft$

Lemma A. One can choose an orientation K of Γ s.t. $\forall$ face f, $n^K(\partial f)$ is odd.

Proof :



$\Gamma' \supset T$ max tree rooted in outer face

 $\rightarrow$ 

$\square$

Lemma B. If K is s.t. $n^K(\partial f) \equiv 1 \ (2)$ $\forall f$ face, then every simple closed curve $C \subset \Gamma$ oriented $\circlearrowleft$ satisfies $n^K(C) \equiv \#\{\text{vertices enclosed by } C\} + 1 \quad (2)$.

Proof : C closed curve $\subset \Gamma$ planar $\Rightarrow$ C bounds $f_1 \cup \dots \cup f_k$ faces $(k \geq 1)$.

Proof by induction over $k \geq 1$, the start $k=1$ being exactly the assumption $\square$

Lemma C. In $\Gamma \subset \mathbb{R}^2$, any cycle C s.t. $\Gamma \setminus V(C)$ has a perfect match D encloses an even nb of vertices

Proof : They are matched by D.

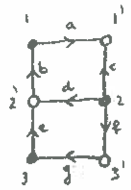
$\square$

Conclusion : $\Gamma \subset \mathbb{R}^2$, K orientation s.t. $n^K(\partial f)$ odd $\forall$ face f $\Rightarrow$ K is pfaffian

$\Rightarrow Z = |\mathrm{Pf}(A^K)|$

$\square$

Examples :

1. 

$$M = \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{pmatrix} -a & -c & 0 \\ b & -d & -e \\ 0 & -f & g \end{pmatrix} & \begin{array}{c} 1' \\ 2' \\ 3' \end{array} \end{array} \quad , \quad \det M = adg + aef + bcg$$

2. (Kasteleyn) : Let Γ be the $m \times n$ square lattice (wlog, m even), with weights $\big| y \quad \xrightarrow{x}$

Then, $$Z_{mn}(x,y) = \prod_{k=1}^{m/2} \prod_{\ell=1}^{n} 2\left(x^2 \cos^2\left(\frac{k\pi}{m+1}\right) + y^2 \cos^2\left(\frac{\ell\pi}{n+1}\right)\right)^{1/2}.$$

$$\Rightarrow \frac{1}{n^2} \log Z_{nn}(x,y) \xrightarrow{n\to\infty} \frac{1}{(2\pi)^2} \int_0^\pi \int_0^\pi \log\left(2\left(x^2+y^2+x^2\cos\varphi + y^2\cos\psi\right)\right) d\varphi\, d\psi \quad \underset{x=y=1}{\Rightarrow} \text{ i.e. } \#\mathcal{D}(\Gamma_{n\times n}) \sim e^{n^2 \cdot G/\pi},$$

$G = 1^{-2} - 3^{-2} + 5^{-2} - 7^{-2} + \dots$ Catalan's constant.

__Question__ : What about general (non-planar) graphs ?

Bad news [Valiant, 1976] : computing $\#\mathcal{D}(\Gamma)$ on an arbitrary graph is a "#P-complete problem"

(i.e. at least as difficult as counting # accepting computations of any non-deterministic polynomial-time Turing machine)

$\Rightarrow$ not only are not all graphs pfaffians, but we don't expect a pol. time alg. in general.

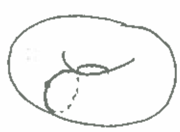
Good news : The method of Kasteleyn can be extended to any finite graph (in an arbitr. surface).

For this, we need some geometric tools...

I.3. Homology and spin structures

- The planarity of Γ was used once in the proof of Kasteleyn's theorem : C cycle $\subset \mathbb{R}^2 \Rightarrow C$ bounds faces.

In general, an arbitrary finite graph Γ can be embedded in a compact surface Σ (s.t. $\Sigma \setminus \Gamma$ = discs),

of some genus $g \geq 0$. And if $g > 0$, the fact above doesn't hold :  $g=1$

But there is a tool to measure how badly the fact above doesn't hold, homology.

Working over $\mathbb{Z}_2$ will make things easy to define, and is all we will need.

Fix $\Gamma \subset \Sigma$ as above, and set $C_0 :=$ $\mathbb{Z}_2$-vector space with basis the vertices of Γ

$C_1 :=$ ——————————— edges of Γ

$C_2 :=$ ——————————— faces of $\Gamma \subset \Sigma$.

Define boundary operators $\partial_1 : C_1 \to C_0$, $\partial_2 : C_2 \to C_1$, extended $\mathbb{Z}_2$-linearly,

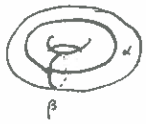
 $\mapsto v + v'$ 

Note that $\partial_1 \circ \partial_2 = 0$, so that $\{\text{boundaries}\} := \mathrm{Im}\,\partial_2 \subset \mathrm{Ker}\,\partial_1 =: \{\text{cycles}\}$.

Define $H_1(\Sigma;\mathbb{Z}_2) := \mathrm{Ker}\,\partial_1 / \mathrm{Im}\,\partial_2$, the first homology group of Σ (over $\mathbb{Z}_2$)

Facts:
- $H_1(\Sigma;\mathbb{Z}_2)$ only depends on Σ: it is a $\mathbb{Z}_2$-v.s of dimension $2g$.
- It admits a non-deg bilinear form: the intersection form

Ex:



$\alpha \cdot \beta = 1$

Finally, we shall call a map $q: H_1(\Sigma;\mathbb{Z}_2) \to \mathbb{Z}_2$ a quadratic form if $q(x+y) = q(x) + q(y) + x \cdot y \quad \forall x,y \in H_1(\Sigma$
The Arf invariant (aka democratic invariant) of q is $\mathrm{Arf}(q) \in \mathbb{Z}_2$ defined by $\mathrm{Arf}(q) =$ the value most taken by q.
The set Q of such quadratic forms is in bijection with $\mathrm{Hom}(H_1(\Sigma;\mathbb{Z}_2);\mathbb{Z}_2) =: H^1(\Sigma;\mathbb{Z}_2)$, i.e., it has 2^{2g} el

- How can one count the rotation number of a closed curve on a surface?

In $\mathbb{R}^2$, there is a unique sensible way to do so ... in Σ, there are several, given by "spin structures".
These can be described by vector fields on Σ with isolated zeros of even index.

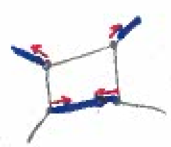
Fact [D. Johnson, 1980]: Fix X a v.f on Σ with zeros of even index.
Given a closed simple curve $C \subset \Sigma$, set $q_X(C) := \left(\begin{array}{l}\text{rotation nb of } X \text{ along } C \text{ w.r} \\ \text{to the velocity vector field of } C\end{array}\right) + 1 \pmod 2 \in \mathbb{Z}_2$
Then, this gives a well-defined quadratic form $q_X \in Q$.

I.4. The Pfaffian formula

(joint work w/ N. Reshetikhin)

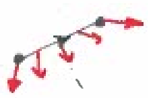
We will now use the tools described above to get a formula for $Z(\Gamma,\nu)$ for any $\Gamma \subset \Sigma$.
So, fix $\Gamma \subset \Sigma$ with $\# V(\Gamma)$ even. We want to construct a vector field X on Σ with zeros of even index

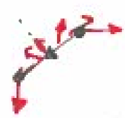
- to construct it along $V(\Gamma)$, fix a dimer conf. $D_0 \in \mathcal{D}(\Gamma)$.



- There are essentially 2 ways to extend it along edges of Γ,

that can be encoded by an orientation of the edge:





- extend it to a v.f X on Σ, with one zero in each face.

Easy fact: the index of the zero of X in the face f has the parity of $n^K(\partial f) + 1$

So the vector field is as we want exactly when $n^K(\partial f)$ is odd $\forall f$. This leads to:

Definition: • An orientation K on a graph $\Gamma \subset \Sigma$ is called a Kasteleyn orientation if $\forall$ face f, $n^K(\partial f)$ is odd.

• Two such orientations are called equivalent if they are related by flipping orientations around vertices.

Ex:  $\subset S^2$ $\sim$ 

Proposition: There exists a Kasteleyn orientation on $\Gamma \subset \Sigma$ $\Leftrightarrow$ $\# V(\Gamma)$ is even.

In such a case, there are 2^{2g} equ. classes of Kast. orient. on $\Gamma \subset \Sigma$. $\square$

given by switching orientations of edges intersecting basis el'ts $\gamma_1 \ldots \gamma_{2g}$ of $H_1(\Sigma)$

Putting all together, given a $D_0 \in \mathcal{D}(\Gamma)$, we have maps:

$$\{\text{Kast. orient. on } \Gamma \subset \Sigma\} \xrightarrow{\text{constr. above}} \{\text{vector fields on } \Sigma\} \xrightarrow{\text{Johnson}} Q$$
$$K \longmapsto q^K_{D_0}$$

and $q^K_{D_0}$ can be computed explicitly: C oriented simple closed curve on Γ $\Rightarrow$ $q^K_{D_0}([C]) = n^K(C) + \ell_D(C) + 1$

where $\ell_D(C)$ = # dimers of D sticking out to the left of C.

And we now know "for free" that this is a well-defined quadratic form on $H_1(\Sigma;\mathbb{Z}_2)$.

Let's use this: fix K Kasteleyn on $\Gamma \subset \Sigma_g$, $A^K = A^K(\Gamma,\nu)$ the corr. Kast. matrix.

$$\overbrace{\varepsilon^K(D_0)}^{\pm 1 \text{ wlog}} \operatorname{Pf}(A^K) = \sum_{D \in \mathcal{D}(\Gamma)} \varepsilon^K(D_0)\varepsilon^K(D) \cdot \nu(D) \qquad D \Delta D_0 = \coprod_i C_i$$

$$= \sum_{D \in \mathcal{D}(\Gamma)} \Big(\prod_i (-1)^{n^K(C_i)+1}\Big) \nu(D) \qquad q^K_{D_0}(C_i) = n^K(C_i) + \overbrace{\ell_{D_0}(C_i)}^{=0} + 1 = n^K(C_i) + 1$$

$$= \sum_{D \in \mathcal{D}(\Gamma)} (-1)^{\sum_i q^K_{D_0}(C_i)} \nu(D) = \sum_{D \in \mathcal{D}(\Gamma)} (-1)^{q^K_{D_0}([D \Delta D_0])} \nu(D)$$

$$= \sum_{\alpha \in H_1(\Sigma;\mathbb{Z}_2)} (-1)^{q^K_{D_0}(\alpha)} Z_\alpha(D_0), \qquad Z_\alpha(D_0) = \sum_{\substack{D \in \mathcal{D}(\Gamma) \\ [D \Delta D_0] = \alpha}} \nu(D)$$

2^{2g} eqn., 2^{2g} unknown $(Z_\alpha$'s$)$: can be solved and summed over $\sum_\alpha Z_\alpha = Z$ to get:

Theorem:

$$\boxed{Z(\Gamma,\nu) = \frac{1}{2^g} \sum_{[K]} (-1)^{Arf(q^K_\omega)}\, Pf(A^K)}$$

Examples: 1. $g=0 \Rightarrow Z = Pf(A^K)$

2. $g=1 \Rightarrow Z = \frac{1}{2}\left(-Pf(A^{K_{00}}) + Pf(A^{K_{11}}) + Pf(A^{K_{01}}) + Pf(A^{K_{10}})\right)$ for K_{00} well-chosen.

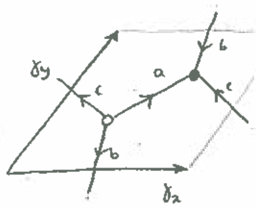
I.5. Bipartite graphs on the torus

(following Kenyon-Okounkov-Sheffield)

Now, let us assume that $\Gamma \subset \mathbb{T}^2$ is bipartite (with $\#B = \#W$), and fix a Kast. orient. on $\Gamma \subset \mathbb{T}^2$.

Recall that $A^K = \begin{array}{cc} \begin{array}{cc} W & B \end{array} & \\ \begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix} & \begin{array}{l} \}W \\ \}B \end{array} \end{array}$, so $Pf(A^K) = \pm \det M$.

Ex:



Let's modify M as follows: Pick a basis γ_x, γ_y of $H_1(\mathbb{T}^2)$ transverse to Γ,

and multiply the coeff of edge as follows 

$P(z,w) = a + \frac{b}{z} + cw$

This gives $M(z,w)$, $z,w \in \mathbb{C}^*$.

Write $P(z,w) = \det M(z,w) \in \mathbb{R}[z^{\pm1}, w^{\pm1}]$: the characteristic polynomial of Γ

Note that the Pfaffian formula now reads:

$$Z = \frac{1}{2}\left(-P(1,1) + P(-1,1) + P(1,-1) + P(-1,-1)\right),$$

for a well chosen Kast. orientation.

Theorem [KOS]:

For any weighted bipartite graph $(\Gamma,\nu) \subset \mathbb{T}^2$, the associated spectral curve $\{(z,w) \in (\mathbb{C}^*)^2 \mid P(z,w) = 0\} \subset (\mathbb{C}^*)^2$ is a "special Harnack curve", i.e.: it meets each torus $r S^1 \times s S^1 := \{(z,w) \in (\mathbb{C}^*)^2 \mid |z| = r, |w| = s\}$ at most twice.

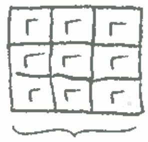
This seemingly technical result is extremely deep, and provides all the necessary information for the rigorous study of large scale properties of the dimer model on such graphs. We will also use it for the Ising model.

Before that, I will just mention one application by KOS.

As often, it is interesting to look at the model when the graph gets bigger and bigger.

For $\Gamma \subset \mathbb{T}^2$, there is an obvious natural way to do so: if $\Gamma = \boxed{\Gamma}$,

write Γ_n for



$\Big\}$ n copies $\subset \mathbb{T}^2$.

n copies

Then, the <u>free energy per fundamental domain</u> is defined by $\log Z := \lim\limits_{n\to\infty} \frac{1}{n^2} \log Z(\Gamma_n)$. (fct of ν).

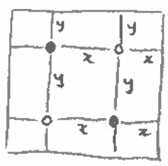
Ex: Setting $\nu \equiv 1$, we have $\#\mathcal{D}(\Gamma_n) \sim e^{n^2 \cdot \log Z|_{\nu=1}}$ (limit exist by subadditivity)

Theorem [KOS]:

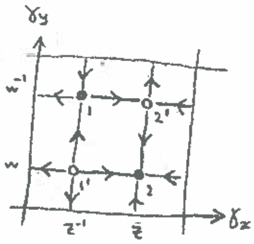
For any weighted bipartite graph Γ on the torus,

$$\log Z = \int_0^1\int_0^1 \log |P(e^{2\pi i \varphi}, e^{2\pi i \psi})|\, d\varphi\, d\psi = \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2} \log|P(z,w)| \frac{dz}{z}\frac{dw}{w}.$$

Example. $\Gamma =$



$\rightsquigarrow$



$$M(z,w) = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ & \begin{pmatrix} y(1+z^{-1}) & x(1+w) \\ -x(1+w^{-1}) & y(1+z) \end{pmatrix} \begin{array}{c} 1' \\ 2' \end{array} \end{array}$$

$$\Rightarrow \quad P(z,w) = y^2(2+(z+z^{-1})) + x^2(2+(w+w^{-1}))$$

$$\Rightarrow \quad P(e^{2\pi i\varphi}, e^{2\pi i \psi}) = 2\cdot\left(x^2+y^2 + x^2\cos(2\pi\psi) + y^2\cos(2\pi\varphi)\right)$$

$$\Rightarrow \quad \log Z = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \log\left(2\left(x^2+y^2+x^2\cos\psi + y^2\cos\varphi\right)\right) d\varphi\, d\psi = 4 \times (\text{the result of Kasteleyn}) \quad (\text{fund. domain } 4\times$$

<u>Idea of the proof:</u>

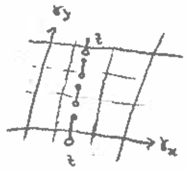
1. Writing P_n for the characteristic polynomial of Γ_n, and $Z_n^{\theta\tau} := P_n((-1)^\theta, (-1)^\tau)$, we have

$$\forall\, \theta,\tau, \quad |Z_n^{\theta\tau}| \overset{-\text{ signs}}{\leq} Z(\Gamma_n) = \frac{1}{2}\left(-Z_n^{00} + Z_n^{10} + Z_n^{01} + Z_n^{11}\right) \leq 2 \max_{(\theta,\tau)} |Z_n^{\theta,\tau}|$$

$$\Rightarrow \quad \log Z := \lim_{n\to\infty}\frac{1}{n^2}\log Z(\Gamma_n) = \lim_{n\to\infty}\frac{1}{n^2}\log\left(\max_{\theta,\tau}|Z_n^{\theta,\tau}|\right) = \lim_{n\to\infty}\frac{1}{n^2}\log\left(\max_{(\theta,\tau)}|P_n((-1)^\theta,(-1)^\tau)|\right)$$

2. $$P_n(z,w) = \prod_{u^n=z}\ \prod_{v^n=w} P_1(u,v)$$

[

$P_n(z,w) = \det \mathcal{M}_n(z,w)$ is unchanged if one replaces  by 

$$\mathbb{C}^B = \bigoplus_{\substack{\alpha^n=1\\ \beta^n=1}} V_B^{\alpha,\beta}, \qquad V_B^{\alpha,\beta} = \{ f \in \mathbb{C}^B \mid f(b+\tau_1) = \alpha\cdot f(b),\ f(b+\tau_2) = \beta\cdot f(b)\}$$

idem $\mathbb{C}^W$ $$\Rightarrow \quad \widetilde{\mathcal{M}}_n(z,w) = \bigoplus_{\substack{\alpha^n=1\\ \beta^n=1}} \left(\widetilde{\mathcal{M}}_n\big|: V_B^{\alpha,\beta} \to V_W^{\alpha,\beta}\right) = \bigoplus_{\substack{\alpha^n=1\\ \beta^n=1}} \mathcal{M}_1(\alpha\cdot z^{1/n}, \beta\cdot z^{1/n}).$$

]

3. For $\theta,\tau \in \{0,1\}$, we get

$$\frac{1}{n^2}\log|P_n((-1)^\theta,(-1)^\tau)| = \frac{1}{n^2}\sum_{u^n=(-1)^\theta}\ \sum_{v^n=(-1)^\tau} \log|P(u,v)|, \quad \text{Riemann sum for } \int_0^1\!\!\int_0^1 \log|P(e^{2\pi i\varphi}, e^{2\pi i\psi})|\,d\varphi\, d\psi$$

By 1., we just need to show that for at least one choice of θ,τ, this integral converges. (if it diverges, it's smaller than mean

But by the previous thm, $P(z,w)$ has at most 2 (conj.) zeroes on $\mathbb{T}^2$.

Since the Riemann sums above (for different θ,τ) are on staggered lattices, at least 3 out of 4 of them converge.

[

— end of Part I —