Galactic dynamics – Problem set 5. Spring 2021

The answers should be returned by ***Tuesday (13.4) 4pm (16.00) in Moodle****, link through the official course homepage. The answers to the problem set will be discussed on Thursday (15.4) at 14.15-16.00 on Zoom.*

1. Starting from the collisionless Boltzmann equation in cylindrical coordinates, expressed in velocities:
$$\frac{\partial f}{\partial t}+v_R\frac{\partial f}{\partial R}+\frac{v_\phi}{R}\frac{\partial f}{\partial \phi}+v_z\frac{\partial f}{\partial z}+\left[\frac{v_\phi^2}{R}-\frac{\partial \Phi}{\partial R}\right]\frac{\partial f}{\partial v_R}-\frac{1}{R}\left[v_R v_\phi+\frac{\partial \Phi}{\partial \phi}\right]\frac{\partial f}{\partial v_\phi}-\frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial v_z}=0,$$
derive the v_z-component Jeans equation (Eq. 4.222b),
$$\frac{1}{R}\frac{\partial(R\nu\overline{v_R v_z})}{\partial R}+\frac{\partial(\nu\overline{v_z^2})}{\partial z}+\nu\frac{\partial \Phi}{\partial z}=0$$
by multiplying the collisionless Boltzmann equation by v_z and integrating over the velocity space for an axisymmetric system, so that all derivatives with respect to t and ϕ vanish.

2. Derive next the v_ϕ-component Jeans equation (eq. 4.222c),
$$\frac{1}{R^2}\frac{\partial(R^2\nu\overline{v_R v_\phi})}{\partial R}+\frac{\partial(\nu\overline{v_z v_\phi})}{\partial z}=0$$
by multiplying the collisionless Boltzmann equation in Problem 1 by v_ϕ and integrating over the velocity space for an axisymmetric system, so that all derivatives with respect to t and ϕ vanish.

3. Show that in the presence of an externally generated gravitational potential $\Phi_{\rm ext}$ the right side of equation (4.247) acquires an extra term:
$$V_{jk}=-\frac{1}{2}\int d^3\vec{x}\left(x_k\frac{\partial \Phi_{\rm ext}}{\partial x_j}+x_j\frac{\partial \Phi_{\rm ext}}{\partial x_k}\right)\rho$$

4. For over 150 years, most astronomers believed that Saturn's rings were rigid bodies, until Laplace showed that a solid ring would be unstable. The same instability plagues Larry Niven's famous fictional planet Ringworld. Following Laplace, consider a rigid, circular wire of radius R and mass m that lies in the $x-y$ plane, centered on a planet of mass $M \gg m$ at the origin. The wire rotates around the planet in the $x-y$ plane at the Keplerian angular speed $\Omega_K=(GM/R^3)^{1/3}$. Show that this configuration is linearly unstable and find the growth rate of the instability.

5. We may study the vertical structure of a thin axisymmetric disk by neglecting all radial derivatives and assuming that all quantities vary only in the coordinate z normal to the disk. Thus we adopt the form $f=f(E_z)$ for the distribution function (DF), where $E_z=\frac{1}{2}v_z^2+\Phi(z)$. Show that for an isothermal disk in which $f=\rho_0(2\pi\sigma_z^2)^{-1/2}\exp\left(-E_z/\sigma_z^2\right)$, the approximate form (Eq 2.74 in the lecture notes) of Poisson's equation may be written as:
$$2\frac{d^2\phi}{d\zeta^2}=e^{-\phi},\quad \text{where}\ \phi=\frac{\Phi}{\sigma_z^2},\ \zeta=\frac{z}{z_0},\ \text{and}\ z_0=\frac{\sigma_z}{\sqrt{8\pi G\rho_0}}$$
By solving this equation subject to the boundary conditions $\phi(0)=\phi'(0)=0$, show that the density in the disk is given by (Spitzer 1942)
$$\rho(z)=\rho_0\text{sech}^2\left(\frac{z}{2z_0}\right).$$
Finally, show that the surface density of the disk is:
$$\Sigma=\frac{\sigma_z^2}{2\pi G z_0}=4\rho_0 z_0.$$