Galaxy formation and evolution equations

- Sérsic profile: $I(R) = I_e e^{-\beta_n[(R/R_e)^{1/n}-1]}$
  $L = 2\pi \int_0^\infty I(R) R dR = \frac{2\pi n \Gamma(2n)}{(\beta_n)^{2n}} I_0 R_e^2$
- Disk galaxy profile: $I(R) = I_0 e^{-R/R_d}, \quad I_0 = \frac{L}{2\pi R_d^2}$
- Schechter function: $\phi(L)dL = \phi^* \left(\frac{L}{L^*}\right)^\alpha e^{-(L/L^*)} \frac{dL}{L^*}$
- Robertson-Walker metric: $ds^2 = c^2dt^2 - dl^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\vartheta^2 + \sin^2\vartheta d\phi^2)\right]$
- Angular diameter distance: $d_A = \frac{a_0 r}{1+z}$
  Luminosity distance: $d_L = a_0 r(1+z) = d_A(1+z)^2$
- Einstein field equation: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4} T_{\mu\nu}$
- Friedmann equation I: $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{P}{c^2}\right) + \frac{\Lambda c^2}{3}$
  Friedmann equation II: $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}$
- Hubble constant: $H(z) = \left(\frac{\dot{a}}{a}\right)(z) = H_0 E(z)$
  $E(z) = [\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + (1-\Omega_0)(1+z)^2 + \Omega_{\Lambda,0}]^{1/2}$
- Age of the Universe: $t(z) = \int_0^{a(z)} \frac{da}{\dot{a}} = \frac{1}{H_0}\int_z^\infty \frac{dz}{(1+z)E(z)}$
- Angular diameter distance in comoving coordinates: $r = f_K\left[\frac{c}{H_0 a_0}\int_0^z \frac{dz}{E(z)}\right]$
  $f_K(\chi) = \sin\chi \ (K=+1); \ f_K(\chi) = \chi \ (K=0); \ f_K(\chi) = \sinh\chi \ (K=-1)$
- Mattig's formula: $a_0 r = \frac{2c}{H_0}\frac{\Omega_0 z + (2-\Omega_0)[1-(\Omega_0 z+1)^{1/2}]}{\Omega_0^2(1+z)}$
- Gas dynamics: Equation of continuity: $\frac{d\rho}{dt} = -\rho\nabla\cdot\vec{v}$
  Equation of motion: $\frac{d\vec{v}}{dt} = -\frac{1}{\rho}\nabla p - \nabla\phi$
  Gravitational potential: $\nabla^2\phi = 4\pi G\rho$
- Growth of density perturbations: $\frac{d^2\Delta}{dt^2} + 2\left(\frac{\dot{a}}{a}\right)\frac{d\Delta}{dt} = \Delta(4\pi G\rho_0 - k^2c_s^2)$
  General solution: $\Delta(a) = \frac{5\Omega_0}{2}\left(\frac{1}{a}\frac{da}{dt}\right)\int_0^a \frac{da'}{(da'/dt)^3}$
- Evolution of peculiar velocites: $\frac{d\vec{u}}{dt} + 2\left(\frac{\dot{a}}{a}\right)\vec{u} = -\frac{1}{a^2}\nabla_c\delta\phi$
- Relativistic case: $\frac{d^2\Delta}{dt^2} + 2\left(\frac{\dot{a}}{a}\right)\frac{d\Delta}{dt} = \Delta\left(\frac{32\pi G\rho}{3} - k^2c_s^2\right)$
- Perturbed metric: $ds^2 = a^2(\tau)\{(1+2\phi)d\tau^2 + 2w_i d\tau dx^i - [(1-2\psi)\gamma_{ij} + 2h_{ij}]dx^i dx^j\}$
- The sound speed: $c_S^2 = \frac{(\partial p/\partial T)_r}{(\partial\rho/\partial T)_r + (\partial\rho/\partial T)_m} = \frac{c^2}{3}\frac{4\rho_r}{4\rho_r + 3\rho_m}$
- Instabilities in the presence of dark matter:

$$\begin{cases} \ddot{\Delta}_B + 2\left(\frac{\dot{a}}{a}\right)\dot{\Delta}_B = 4\pi G\rho_B\Delta_B + 4\pi G\rho_D\Delta_D \\ \ddot{\Delta}_D + 2\left(\frac{\dot{a}}{a}\right)\dot{\Delta}_D = 4\pi G\rho_B\Delta_B + 4\pi G\rho_D\Delta_D \end{cases}$$

- Two-point correlation function and the power spectrum: $dN(r) = N_0[1+\xi(r)]dV$
  $\xi(r) = \frac{V}{2\pi^2}\int |\Delta_k|^2 \frac{\sin kr}{kr} k^2 dk = \frac{V}{2\pi^2}\int P(k)\frac{\sin kr}{kr}k^2 dk$
  $P(k) = \frac{1}{V}\int_0^\infty \xi(r)\frac{\sin kr}{kr}4\pi r^2 dr$
- Transfer function: $\Delta_k(z=0) = T(k)f(z)\Delta_k(z)$
- Parametric solution: $a_p = A(1-\cos\theta)\quad t = B(\theta - \sin\theta)$
  $A = \frac{\Omega_0}{2(\Omega_0-1)}\quad B = \frac{\Omega_0}{2H_0(\Omega_0-1)^{3/2}}$
- Press-Schechter mass function: $N(M) = \frac{1}{2\sqrt{\pi}}\left(1+\frac{n}{3}\right)\frac{\bar{\rho}}{M^2}\left(\frac{M}{M^*}\right)^{(3+n)/6}\exp\left[-\left(\frac{M}{M^*}\right)^{(3+n)/3}\right]$
- Navarro-Frenk-White profile: $\frac{\rho(r)}{\rho_{\rm crit}} = \frac{\delta_c}{(r/r_s)(1+r/r_s)^2}$
- Virial temperature: $T_{\rm vir} = \frac{\mu m_p}{2k_B}v_{\rm vir}^2 \simeq 3.6\times10^5{\rm K}\left(\frac{\rm v_{vir}}{100\ {\rm kms^{-1}}}\right)^2$
- Cooling time: $t_{\rm cool} = \frac{3nk_BT}{2n_H^2\Lambda(T)} \simeq 3.3\times10^9{\rm yr}\left(\frac{\rm T}{10^6{\rm K}}\right)\left(\frac{\rm n}{10^{-3}{\rm cm^{-3}}}\right)^{-1}\left(\frac{\Lambda({\rm T})}{10^{-23}{\rm ergs^{-1}cm^3}}\right)^{-1}$
- Free-fall time: $t_{\rm ff} = \left(\frac{3\pi}{32G\rho}\right)^{1/2} \simeq 3.6\times10^6\ {\rm yrs}\left(\frac{\rm n_{H_2}}{100\ {\rm cm^{-3}}}\right)^{-1/2}$
- Galaxy disk mass: $M_d = m_d M_{\rm vir} \simeq 1.3\times10^{11}h^{-1}M_\odot\left(\frac{m_d}{0.05}\right)\left(\frac{v_{\rm vir}}{200\ {\rm km/s}}\right)^3 D^{-1}(z)$
  $D(z) = \left[\frac{\Delta_{\rm vir}(z)}{100}\right]^{1/2}\left[\frac{H(z)}{H_0}\right]$
- Spin parameter: $\lambda = \frac{J_{\rm vir}|E|^{1/2}}{GM_{\rm vir}^{5/2}} = \frac{1}{j_d}\frac{J_d|E|^{1/2}}{GM_{\rm vir}^{5/2}}$
- Impulse approximation: $\Delta E_S = \frac{4}{3}G^2M_S\left(\frac{M_P}{v_P}\right)^2\frac{\langle r^2\rangle}{b^4}$
- Dynamical friction: $\vec{F}_{\rm df} = M_S\frac{d\vec{v}_S}{dt} = -4\pi\left(\frac{GM_S}{v_S}\right)^2\ln\Lambda\rho(<v_S)\frac{\vec{v}_S}{v_S}$
  $F_{\rm df} = -0.428\frac{GM_S^2}{r^2}\ln\Lambda\frac{\vec{v}_S}{v_S}$
  $t_{\rm df} \simeq 0.117\frac{(M_h/M_s)}{\ln(M_h/M_s)}t_H$
- Jeans equation: $\frac{1}{\rho}\frac{d(\rho\langle v_r^2\rangle)}{dr} + 2\beta\frac{\langle v_r^2\rangle}{r} = -\frac{d\Phi}{dr},\qquad \beta(r) = 1-\langle v_\vartheta^2\rangle/\langle v_r^2\rangle$
- Supermassive black holes: $r_{\rm BH} = \frac{GM_{\rm BH}}{\sigma^2} = 10.8{\rm pc}\left(\frac{M_{\rm BH}}{10^8M_\odot}\right)\left(\frac{\sigma}{200{\rm kms^{-1}}}\right)^{-2}$
  $\log(M_{\rm BH}/M_\odot) = 8.12 + 4.24\log(\sigma/200\ {\rm kms^{-1}})$
- Eddington luminosity: $L_{\rm edd} = \frac{4\pi Gcm_p}{\sigma_T}M_{\rm BH} \approx 1.28\times10^{46}\left(\frac{M_{\rm BH}}{10^8M_\odot}\right)\ {\rm ergs^{-1}}$
- Synchrotron emission: $\langle P_S\rangle = \frac{4}{3}\sigma_Tc\gamma^2\beta^2u_B$
  Inverse Compton emission: $\langle P_{IC}\rangle = \frac{4}{3}\sigma_Tc\gamma^2\beta^2u_{\rm rad}$
- Growth of supermassive black holes: $\dot{M}_{\rm BH} = \frac{L}{\epsilon_rc^2} = \left(\frac{L}{L_{\rm edd}}\right)\frac{M_{\rm BH}}{\epsilon_rt_{\rm edd}}$
  $t_{\rm Edd} = \frac{\sigma_Tc}{4\pi Gm_p} \approx 4.4\times10^8\ {\rm yr}$
- AGNs and galaxy formation: $\frac{E}{|W|} \sim \frac{\bar{\epsilon}M_{\rm BH}}{M_{\rm gal}}\left(\frac{c}{\sigma}\right)^2$