SOME RECENT RESULTS ON THE 
SCHRAMM-LOEWNER EVOLUTION 
(SLE)

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SCHRAMM-LOEWNER EVOLUTION
\((SLE_\kappa)\)

- Measure \(\mu_D(z, w)\) on paths \(\gamma\) connecting distinct points of a domain \(D \subset \mathbb{C}\).

- The endpoints \(z, w\) can be boundary points or interior (bulk) points.

- The total mass \(\Psi_D(z, w) = \|\mu_D(z, w)\|\) is called the \textit{partition function}. It is expected to be the normalized limit of discrete partition functions.
• If $0 < \Psi_D(z, w) < \infty$, we can define a probability measure

$$\mu_D^\#(z, w) = \frac{\mu_D(z, w)}{\Psi_D(z, w)}.$$ 

• The measure will be supported on curves of a particular fractal dimension that we denote $d$.

$$d = \min \left\{ 1 + \frac{\kappa}{8}, 2 \right\}.$$
• How does the measure change under conformal transformation?

• How does the measure change under perturbation of the domain?

• What is the conditional distribution of the curve given some part of the curve?

• What is the probability of getting near to a point $\zeta \in D$?
We expect that the curve is parametrized by the normalized version of the length of the curve. Suppose

\[ f : D \rightarrow f(D) \]

is a conformal transformation. Since the curve has fractal dimension \( d \), the time to traverse \( f(\gamma[s, t]) \) should be

\[ \int_s^t |f'(\gamma(r))|^d \, dr. \]

- If \( \mu \) is a measure on curves from \( z \) to \( w \) in \( D \), let Define \( f \circ \mu \) by

\[ f \circ \mu[E] = \mu\{\gamma : f \circ \gamma \in E\}. \]
CONFORMAL COVARIANCE

• The conformal covariance rule is

\[ f \circ \mu_D(z, w) = \left| f'(z) \right|^{b_z} \left| f'(w) \right|^{b_w} \mu_{f(D)}(f(z), f(w)), \]

where \( b_\zeta = b \) if \( \zeta \in \partial D \) and \( b_\zeta = \tilde{b} \) if \( \zeta \in D \)

where

\[ b = \frac{6 - \kappa}{2\kappa}, \quad \tilde{b} = \frac{b(\kappa - 2)}{4}. \]

This assumes smoothness at boundary points.

• The probability measures are conformally invariant

\[ f \circ \mu_D^\#(z, w) = \mu_{f(D)}^\#(f(z), f(w)). \]

The measure \( \mu_D^\#(z, w) \) can be defined even if the boundary is not smooth.
SCHRAMM’S CONSTRUCTION

- $D$ simply connected, $z \in \partial D$, $w \in D \cup \partial D$.

- Constructed $\mu^\#_D(z, w)$ as a measure on curves *modulo reparametrization*. Used a capacity parametrization.

- Assumptions were conformal invariance and *domain Markov property*: given an initial configuration $\gamma_t = \gamma[0, t]$ the distribution of the remainder of the path is

$$\mu^\#_{D\setminus \gamma_t}(\gamma(t), w).$$
• There is a one-parameter family of curves, indexed by $\kappa$ with fractal dimension (Rohde-S, Beffara)

$$d = \min \left\{ 1 + \frac{\kappa}{8}, 2 \right\}.$$ 

The curves are simple for $\kappa \leq 4$ and plane-filling for $\kappa \geq 8$.

• Reparametrized the curve so that the conformal map satisfies a stochastic differential equation. This equation is the basis for most analysis.
PARTITION FUNCTION FOR $SLE_\kappa$ FOR SIMPLY CONNECTED DOMAINS

- The partition function for $SLE_\kappa$ is normalized so that $\Psi_\mathbb{H}(0, \infty) = \Psi_\mathbb{D}(0, 1) = \Psi_\mathbb{H}(0, 1) = \Psi_\mathbb{C}(0, \infty) = 1$.

- Scaling rule

$$\Psi_D(z, w) = |f'(z)|^{b_z} |f'(w)|^{b_w} \Psi_{f(D)}(f(z), f(w)),$$

where $b_\zeta$ is either the boundary scaling exponent $b = (6 - \kappa)/(2\kappa)$ or the interior scaling exponent $\tilde{b} = b(\kappa - 4)/2$. In particular,

$$\Psi_\mathbb{H}(0, x) = |x|^{-2b}.$$
GENERALIZED RESTRICTION
(L. - S. - Werner)

• The values of the scaling exponent are obtained from computations on Schramm's original process.

• Suppose simply connected $D = \mathbb{H} \setminus K$ with $K$ compact and away from 0. Let $z \in \overline{D}$, and $\tau$ a stopping time for an $SLE$ path such that $\gamma_\tau = \gamma(0, \tau]$ does not hit $\mathbb{H} \cap \partial D$. 
• Let $\mu = \mu_H(0, \infty), \hat{\mu} = \mu_D(0, z)$. For small $\tau$, considered as measures on $\gamma_\tau$, the two measures (on paths modulo reparametrization) are absolutely continuous with Radon-Nikodym derivative

$$\frac{d\hat{\mu}}{d\mu}(\gamma_\tau) = \frac{\Psi_D(\gamma(\tau), z)}{\Psi_H(\gamma(\tau), \infty)} \exp\left\{ \frac{c}{2} \Lambda_H(\gamma_\tau, \mathbb{H} \setminus D) \right\}.$$

• Here $D_\tau$ is the connected component of $D \setminus \gamma_\tau$ containing $z$, and $c$ is the central charge

$$c = b(3\kappa - 8) = \frac{(6 - \kappa)(3\kappa - 8)}{2\kappa}.$$
• Although the partition functions in the ratio
\[ \frac{\Psi_{D_{\tau}}(\gamma(\tau), z)}{\Psi_{\mathbb{H}_{\tau}}(\gamma(\tau), \infty)} \]
do not exist, the ratio is well defined.

• Let \( g : \mathbb{H}_{\tau} \rightarrow \mathbb{H} \) be a conformal transformation with \( g(\infty) = \infty, g'(\infty) = 1 \). Then formally we can write the ratio as
\[ \frac{|g'(\gamma(\tau))|^b |g'(z)|^{b_z} \Psi_{g(D_{\tau})}(g(\gamma(\tau)), g(z))}{|g'(\gamma(\tau))|^b |g'(\infty)|^{b} \Psi_{\mathbb{H}}(g(\gamma(\tau)), \infty)} \]

• Cancelling the infinite term, and using \( g'(\infty) = 1, \Psi_{\mathbb{H}}(x, \infty) = 1 \), this gives
\[ |g'(z)|^{b_z} \Psi_{g(D_{\tau})}(g(\gamma(\tau)), g(z)). \]
• If \( z = \infty \), then this term equals

\[
\Psi_{g(\tau)}(g(\gamma(\tau)), \infty).
\]

For \( \kappa \leq 4 \), we can let \( \tau = \infty \) and write

\[
\frac{d\tilde{\mu}}{d\mu}(\gamma_\tau) = \frac{\Psi_{D\tau}(\gamma(\tau), z)}{\Psi_{H\tau}(\gamma(\tau), \infty)} \exp \left\{ \frac{c}{2} \Lambda_{H}(\gamma, H \setminus D) \right\}.
\]
BROWNIAN LOOP MEASURE (L.-W.)

- Infinite ($\sigma$-finite) measure $\nu_D$ on unrooted, intersecting, loops in $D$.

- Characterized up to multiplicative constant by two properties:
  
  - **Restriction property.** If $D \subset \mathbb{C}$, the measure $\nu_D$ is obtained by restricting $\nu_\mathbb{C}$ to loops that lie in $D$.
  
  - **Conformal invariance.** If $f : D \to f(D)$ is a conformal transformation, then
    
    $$\nu_{f(D)} = \nu_D$$

- $\Lambda_D(V, V')$ is the measure on the set of loops in $D$ that intersect both $V$ and $V'$. 
**GREEN’S FUNCTION**

The Green’s function $G_D(\zeta; z, w)$ for $\kappa < 8$ is the $\mu^#_D(z, w)$ probability that a path goes through (near) $\zeta \in D$.

- It satisfies the scaling rule

  $$G_D(\zeta; z, w) = |f'(\zeta)|^{2-d} G_{f(D)}(f(\zeta); f(z), f(w)).$$

- Characterized (up to multiplicative constant) by the fact that

  $$M_t = G_{D \setminus \gamma(t)}(\zeta; \gamma(t), w)$$

  is a local martingale.

- (Rohde-Schramm)

  $$G_{\mathbb{H}}(\zeta; 0, \infty) = [\text{Im } \zeta]^{d-2} [\sin \text{arg } \zeta]^{\frac{8}{\kappa} - 1}.$$
Let $\gamma_D(z)$ denote one-half times the conformal radius of $z$ in $D$, that is,

$$\gamma_{\mathbb{H}}(z) = \text{Im}(z),$$

$$\gamma_{f(D)}(f(z)) = |f'(z)| \gamma_D(z).$$

$$\frac{\text{dist}(z, \partial D)}{2} \leq \gamma_D(z) \leq 2 \text{dist}(z, \partial D).$$

**Theorem:** As $\epsilon \downarrow 0$,

$$P\{\gamma_{D\backslash\gamma_{\infty}}(\zeta) < \epsilon\} \sim c_* G_D(\zeta; z, w) \epsilon^{2-d},$$

$$c_* = 2 \left[ \int_0^\pi \sin^{8/\kappa} x \, dx \right]^{-1}.$$
Write $\zeta = e^{2iz}$, $z = x + iy$.

$$G_D(\zeta; 1, 0) = \Phi(\zeta) \gamma_D(\zeta)^{d-2} \left[ \frac{\sinh y \cosh y}{|\sin z|} \right]^{\frac{8}{\kappa} - 1}.$$ 

The extra term $\Phi(\zeta)$ can be written as

$$\Phi(\zeta) = E^* \left[ g'(0)^q \right], \quad \beta = \frac{(4 - \kappa)(\kappa - 8)}{8\kappa}$$

where $E^* = E^*_\zeta$ is an expectation with respect to radial $SLE_\kappa$ conditioned to go through $\zeta$ (and stopped when it reaches $\zeta$). Here $g$ is a conformal transformation

$$g : \mathbb{D} \to \mathbb{D} \setminus \gamma$$

fixing the origin.
• As in chordal case, as $\epsilon \downarrow 0$,

$$P\{\gamma_{\mathbb{D}\setminus\gamma_\infty}(\zeta) < \epsilon\} \sim c_* G_{\mathbb{D}}(\zeta; 1, 0) \epsilon^{2-d}. $$
TWO-POINT GREEN’S FUNCTION

- Consider $SLE_\kappa$ in $\mathbb{H}$ and let $\gamma(\cdot) = \gamma_{\mathbb{H}\setminus\gamma_\infty}(\cdot)$.

- (L.-Werness) There exists a function $G(z, w)$ such that as $\epsilon, \delta \downarrow 0$,
  \[ P\{\gamma(z) < \epsilon, \gamma(w) < \delta\} \sim c^*_2 G(z, w) (\epsilon \delta)^{2-d}. \]

- Can write $G(z, w) = \tilde{G}(z, w) + \tilde{G}(w, z)$ where
  \[ \tilde{G}(z, w) = G(z) E^*_z \left[ G_{\mathbb{H}\setminus\eta}(z, w) \right]. \]
  Here $E^*_z$ indicates an expectation with respect to SLE conditioned to go through $z$ and $\eta$ is the path stopped at $z$.

- Explicit form is unknown, but (L.-W.-Rezaei)
  \[ G(z, w) \asymp q^{d-2} [q \vee \sin \arg w]^{-\beta}, \quad |z| \leq |w| \]
  \[ q = \frac{|w - z|}{|w|}, \quad \beta = \frac{\kappa}{8} + \frac{8}{\kappa} - 2. \]
NATURAL PARAMETRIZATION
(NATURAL LENGTH)

• Would like to parametrize $SLE_{\kappa}$ such that it is the normalized limit of the length.

• Let $D$ be a bounded simply connected domain with distinct boundary points $z, w$. Let $\gamma$ be an $SLE_{\kappa}$ from $z$ to $w$ (with any increasing continuous parametrization).

• Let $\Theta_t$ denote the “natural length” of $\gamma[0, t]$. (The curve is in the natural parametrization if $\Theta_t = t$.)

• Expect (up to multiplicative constant) that

$$\Theta_\infty = \int_D G_D(\zeta; z, w) \, dA(\zeta).$$
\[ E[\Theta_\infty \mid \gamma_t] = \Theta_t + E[\Theta_\infty - \Theta_t \mid \gamma_t]. \]

\[ E[\Theta_\infty - \Theta_t \mid \gamma_t] = \int G_{D\setminus\gamma_t}(\zeta; \gamma(t), z) \, dA(z). \]

- \( \Theta_t \) is defined to the unique increasing process such that
  \[ \Theta_t + \int G_{D\setminus\gamma_t}(\zeta; \gamma(t), z) \, dA(z) \]
  is a martingale. (Doob-Meyer decomposition)

- Second moment estimates needed to guarantee that this is nontrivial. Similar to construction and estimates of two-point Green’s function.
• (L-Sheffield) The natural length exists and is nontrivial for $\kappa < 5.0 \cdots$. If $\gamma$ has the (usual) capacity parametrization, then $t \mapsto \Theta_t$ is Hölder continuous.

• (L-Zhou) The natural length exists for $\kappa < 8$. The proof uses the two-point Green’s function.

• (L.-Rezaei) A new proof for $\kappa < 8$ that shows that $t \mapsto \Theta_t$ is Hölder continuous. Moreover, the “length” of a path is independent of the domain. The proof uses the Hölder continuity of $\gamma$ in the capacity parametrization.
**INDEPENDENCE OF DOMAIN**

- Let $D = \mathbb{H} \setminus K$ be a simply connected domain with $K$ compact away from the origin.

- Let $\gamma$ be an $SLE_\kappa$ path from 0 to $\infty$ in $D$.

- The natural length can be computed two different ways:
  - Considering $\gamma$ as a curve in $\mathbb{H}$
  - Using the conformal covariance rule.

- The theorem states that these are the same.
OPEN QUESTIONS ON NATURAL LENGTH

• Show that discrete models parametrized by length converge to SLE with natural param.

• Conjecture: the natural length is given (up to constant) by the $d$-dimensional Minkowski content

$$\Theta_t = c_0 \lim_{\epsilon \downarrow 0} \epsilon^{d-2} \text{Area}\{z : \text{dist}(z, \gamma_t) < \epsilon\}.$$ 

• Rezaei (in preparation) has shown that the Hausdorff $d$-dimensional measure is zero. Can we give Hausdorff measure with a gauge function?

• As a preliminary step, can we show reversibility of natural length?
Conformal invariance and the domain Markov property are not sufficient to characterize \( \mu_D^\#(z, w) \) for multiply connected domains. This is because the slit domain \( D \setminus \gamma_t \) is not conformally equivalent to \( D \).

(Zhan) For simply connected domains \( \mu_D^\#(w, z) \) is the same as the reversal of \( \mu_D^\#(z, w) \). (This has recently been extended to \( 4 < \kappa < 8 \) by Miller-Shefffield. It is not true for \( \kappa > 8 \).)

For \( \kappa \leq 4 \), there is a unique way to define \( \mu_D(z, w) \) for multiply connected domains so that conformal covariance and the generalized restriction (boundary perturbation) rule hold.
• The definition (and Zhan's result) immediately show that the measures $\mu(z, w)$ are reversible.

• The definition allows $z, w$ to be interior or boundary points. If $z, w \in \partial D$, they can be in the same component of $\partial D$ (chordal case) or different components (crossing case).

• The definition does not immediately imply that the total mass $\Psi_D(z, w)$ is finite. This has been proved in the cases of doubly connected domains (annuli) and for $\kappa \leq 8/3$ for all domains ($c \leq 0$). Conjectured to be true for $\kappa \leq 4$. and all domains.
(CROSSING) ANNULUS $SLE_\kappa (\kappa \leq 4)$.

$$A_r = \{ r < |z| < 1 \}, \quad w = e^{-r+i\theta}.$$ 

• This is a measure $\mu_{A_r}(1, w)$ on simple paths connecting 1 and $w$ in $A_r$.

• There exists $c_0$ such that the partition function satisfies

$$\Psi_{A_r}(1, w) \sim c_0 e^{-r(\bar{b}-b)} r^{c/2}, \quad r \to \infty.$$ 

• We can get radial $SLE_\kappa$ (up to multiplicative constant) as a limit

$$\mu_{\mathbb{D}}(1, 0) = \lim_{r \to \infty} e^{r(\bar{b}-b)} r^{-c/2} \mu_{A_r}(1, e^{-r}).$$
RADIAL $SLE_\kappa$ FROM THE INTERIOR POINT

- The probability measure $\mu\#(0,1)$ satisfies the following domain Markov property: given an initial segment $\gamma_t$, the distribution on the remainder is annulus $SLE$ in the domain.

- This was derived in a somewhat different way by Zhan.

- Reversibility: Given an initial and a terminal segment, the remainder is annulus $SLE_\kappa$ in the middle.
• Whole plane $SLE_\kappa$ is the probability measure on paths $\mu = \mu_C(0, \infty)$ from 0 to $\infty$ in $\mathbb{C}$ such that given any initial segment $\gamma_t$ the remainder grows like radial $SLE_\kappa$ from $\gamma(t)$ to $\infty$ in the slit domain.

• Suppose that $D$ is a domain containing the origin with a smooth boundary $\partial D$ and $w \in \partial D$.

• Let $\tau$ be a stopping time such that $\gamma_\tau \subset D$.

• Expect that $\mu_1 := \mu_D(0, w)$, considered as a measure on $\gamma_\tau$, is mutually absolutely continuous with respect to $\mu$. Goal: find $d\mu_1/d\mu$. 
• Using the generalized restriction rule we would like to write \([d\mu_1/d\mu](\gamma_\tau)\) as

\[
\frac{\Psi_{D\smallsetminus\gamma_\tau}(\gamma(\tau), w)}{\Psi_{C\smallsetminus\gamma_\tau}(\gamma(\tau), \infty)} \exp\left\{\frac{c}{2} \Lambda(\gamma_\tau, \partial D; \mathbb{C})\right\}.
\]

• The ratio of partition functions can be calculated by taking \(g : \mathbb{C} \smallsetminus \gamma_\tau \rightarrow \mathbb{C} \smallsetminus \mathbb{D} \).

\[
\frac{|g'(w)|^b \Psi_{g(D\smallsetminus\gamma_\tau)}(g(\gamma(\tau)), g(w))}{g'(\infty)^b}.
\]

The partition function in the numerator is an annulus partition function.

• However, \(\Lambda(\gamma_\tau, \partial D; \mathbb{C}) = \infty\).
• Define

\[ \Lambda^*(V_1, V_2) = \lim_{r \downarrow 0} \Lambda(V_1, V_2; O_r) - \log \log(1/r), \]

where \( O_r = \{|z| > r\}. \) Then this limit exists.

• There exists \( c' \) such that \([d\mu_1/d\mu](\gamma_T)\) equals

\[ c' \frac{\Psi_{D \setminus \gamma_T}(\gamma(\tau), w)}{\Psi_{\mathbb{C} \setminus \gamma_T}(\gamma(\tau), \infty)} \exp \left\{ \frac{c}{2} \Lambda^*(\gamma_T, \partial D) \right\}. \]

• We can think of the exponential as a Wick product of a Brownian loop term.
SUMMARY

We can consider $SLE_\kappa$ as having two ingredients:

- Whole plane $SLE_\kappa$

- Brownian loop measure

All other versions can be obtained either as limits, conditional distributions, or by weighting by a Brownian loop measure.