CONFORMAL INVARIANCE OF DOUBLE-DIMER LOOPS

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Dimer model
Double-Dimer model

Two independent uniform random dimer coverings
Long loops.
Pr(loop surrounding both $p_1, p_2$?)
On a surface, a **finite lamination** is an isotopy class of pairwise disjoint non-contractible, non-peripheral simple closed curves.
Let $z_1, \ldots, z_k$ points in $U$. For each finite lamination $L$ in $U \setminus \{z_1, \ldots, z_k\}$, $\Pr_\epsilon(L)$ converges and depends only on the conformal type of $U \setminus \{z_1, \ldots, z_k\}$. (Ignore peripheral and contractible loops.)
Single dimer Kasteleyn matrix for $\mathbb{Z}^2$

\[
\begin{array}{cccc}
-1 & 1 & -1 & \\
i & -i & i & -i \\
1 & -1 & 1 & \\
i & -i & -i & i \\
-1 & 1 & -1 & \\
i & -i & i & -i \\
1 & -1 & 1 & \\
\end{array}
\]

**Theorem [Kasteleyn]** Let $K : \mathbb{C}^W \rightarrow \mathbb{C}^B$ as above.

$K = (K_{w,b})$ where $K_{w,b} = 0$ if $w, b$ are not adjacent and otherwise $K_{w,b} = \{1, i, -1, -i\}$ according to direction.

Then $|\det K|$ is the number of dimer covers.
A function in the kernel of $K$ is \textit{discrete analytic}.
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\[
K f(w) = f(w + \epsilon) - f(w - \epsilon) + i(f(w + i\epsilon) - f(w - i\epsilon))
\]

\[
= 2\epsilon \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f(w)
\]

\[
= 2\epsilon \frac{\partial}{\partial \bar{z}} f(w)
\]

A function in the kernel of $K$ is discrete analytic.

$K^{-1}(b, \cdot)$ is discrete meromorphic with pole at $b$. 
Double dimer model

Let \( \mathbb{K} = \begin{pmatrix} 0 & K \\ K^t & 0 \end{pmatrix} \). (indexed by all vertices).

Then \( \det \mathbb{K} \) is the partition function of double-dimer configurations.

Let \( \Omega(G) \) be the set of “double-dimer configurations”: coverings of \( G \) with even-length loops and doubled edges.

Each configuration in \( \Omega \) with \( k \) nontrivial loops comes from \( 2^k \) pairs of dimer covers.
Double dimer model

Now introduce quaternionic (instead of positive real) edge weights.

$$q = a_0 + a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k} = \begin{pmatrix} a_0 + a_1 i & a_2 + a_3 i \\ -a_2 + a_3 i & a_0 - a_1 i \end{pmatrix}.$$ 

$$q^* = a_0 - a_1 \hat{i} - a_2 \hat{j} - a_3 \hat{k}$$

The weight of $$\omega \in \Omega$$ is now defined to be

$$\prod_{\text{cycles}} (m + m^*)$$

where $$m$$ is the product of the edge weights around the cycle.

Note $$(q_1q_2 \ldots q_k)^* = q_k^* \ldots q_1^*.$$ 

Doubled edges count $$qq^*$$.

This *couples* the individual dimer covers: they are no longer independent.
\[ q_1 q_2^* + q_2 q_1^* \]
\[ (q_1 + q_1^*) q_2 q_2^* \]
\[ (q_2 + q_2^*) q_1 q_1^* \]
\[ q_1 q_1^* q_2 q_2^* \]
\[ q_1 q_1^* \]
\[ q_2 q_2^* \]
Theorem: \( Q_{\text{det}} K = \sum_{\Omega} \prod_{\text{cycles}} (m + m^*) \).

Here \( Q_{\text{det}} \) is the quaternion determinant [Dyson].

It is defined for a quaternion-Hermitian matrix by

\[
Q_{\text{det}} M = \sum_{\sigma \in S_n} (-1)^\sigma \prod_{\text{cycles c of } \sigma} m(c)
\]

Theorem [Mehta]. \( Q_{\text{det}} K_{n \times n} = \sqrt{\det \tilde{K}_{2n \times 2n}} \),

where \( \tilde{K} \) is the matrix obtained by replacing each quaternion with its \( 2 \times 2 \) matrix block.
Example: put weights $q^x$ on north edges:

```
\begin{array}{cccc}
  1 & q & q^2 & q^3 \\
  q & q & q^2 & q^3 \\
  q^2 & q^2 & q^2 & q^3 \\
  q^3 & q^3 & q^3 & q^3 \\
\end{array}
```

Suppose $qq^* = 1$.

Then $Z(q) = \det \mathbb{K}(q)$ counts loops with weight $q^{\text{Area}} + (q^*)^{\text{Area}}$.

In particular $Z(e^{i\theta})$ counts loops with weight $2 \cos(\text{Area} \theta)$. 

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Another example.

Suppose \( qq^* = 1 \).

Loops surrounding \( f_0 \) have weight \( q + q^* \).

All other loops have weight 2.
Another example.

Loops around $A$: $q_1 + q_1^*$

Loops around $B$: $q_2 + q_2^*$

Loops around $A$ and $B$: $q_1 q_2 + (q_1 q_2)^*$

We can choose $q_1, q_2$ so that these three quantities are algebraically independent.
Another example.

Loops around $A$: $q_1 + q_1^* = 2x$

Loops around $B$: $q_2 + q_2^* = 2y$

Loops around $A$ and $B$: $q_1 q_2 + (q_1 q_2)^* = 2z$

We can choose $q_1, q_2$ so that these three quantities are algebraically independent.

$$Z = \sum_{i,j,k \geq 0} C_{i,j,k} x^i y^j z^k$$
**Lemma** (based on [Fock-Goncharov])

By varying the $q$s one can extract from $\det K$ the contribution from any finite lamination.

That is, writing $\det K = \sum_L C_L \prod (m + m^*)$ we have

\[
C_L = \int \phi_L \det K \, dq_1 \ldots dq_k.
\]

This is an integral over $SU(2)^k$.

Can one compute $Z(q) = \det K(q)$?
Theorem:

\[ F(q) := \lim_{\varepsilon \to 0} \frac{Z_\varepsilon(q)}{Z_\varepsilon(1)} \]
exists and is conformally invariant.

Proof idea:

Take a path of weights \( q_t, 0 \leq t \leq 1 \), with \( q_0 = 1 \).

\[ \frac{d}{dt} \log Z_\varepsilon(q_t) = \frac{1}{2} \frac{d}{dt} \log \det \tilde{K}(q_t) \]

which can be written as a sum along the zippers of the Green’s function \( \tilde{K}^{-1}(q_t) \)... and \( \tilde{K}^{-1}(q_t) \) is a discrete meromorphic function.

(depends analytically on the domain).  \( \Box \)
Simple example: $m \times n$ annulus.

Let $q = e^{-n\pi/m}$.

In limit $m, n \to \infty$ with $m/n \to \tau$ (and $m$ even)

$$
\sum_{k=0}^{\infty} \Pr(k \text{ loops}) X^k = \prod_{j=1}^{\infty} \frac{(1 + q^j X + q^{2j})^2}{(1 + q^j + q^{2j})^2}.
$$

0 loops

1 loop

2 loops

...
Take \( q_1 = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \) and \( q_2 = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \).

Then \( Z = \sum_k C_k (2 + t^2)^k \) where \( k \) counts the number of loops surrounding both \( A \) and \( B \).

**Theorem:** \( \mathbb{E}(k) = g(A, B) \), the Dirichlet Green’s function on \( U \).
Uniform cycle-rooted tree (uniform unicycle)

For the uniform unicycle on an $n \times n$ grid,

\[
\mathbb{E}[\text{Area of cycle}] = \frac{2}{\pi} \log n + O(1)
\]
\[
\mathbb{E}[\text{Area}^2] = Cn^2 + o(n^2)
\]

\textbf{Thm}[Levine-Peres]

\[
\mathbb{E}[\text{Length}] = 8 + o(1).
\]
cycles of unicycles on curved surfaces:

CRT on a sphere

CRT on a disk in $\mathbb{H}^2$

(A. Kassel)
intensity of LERW
(joint w/Wilson)

Figure 2. Intensity of loop-erased random walk on $\mathbb{Z}^2$. The origin is at the lower-left, and directed edge-intensities as well as vertex-intensities of the LERW are shown. (See also Figure 11.)
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