SLE(3) variants and the Ising interfaces in multiply-connected domains

Konstantin Izyurov

Chebyshev Laboratory, Saint-Petersburg State University

June 16, 2012
The Ising model

The configurations in the Ising model are all possible assignments of spins \( \pm 1 \) to the lattice sites. The probability of a configuration \( \sigma \) is proportional to

\[
\exp\left[ \frac{1}{T} \sum_{x \sim y} \sigma(x)\sigma(y) \right],
\]

where the sum is taken over all pairs \( x \sim y \) on neighboring sites.
Random curves in the Ising model: interfaces

Contour formulation: instead of pluses and minuses, consider the *interfaces* between them
Random curves in the Ising model: interfaces

Contour formulation: instead of pluses and minuses, consider the *interfaces* between them.
Random curves in the Ising model: interfaces

Contour formulation: instead of pluses and minuses, consider the *interfaces* between them

![Diagram showing a contour with interfaces](image)

The probability of a configuration $S$ is proportional to $\chi^{|S|}$, $\chi = e^{-2/T}$. 
Random curves in the Ising model: interfaces

Contour formulation: instead of pluses and minuses, consider the *interfaces* between them.

![Diagram of random curves]

- $z$
- $a$
Study the *random curves* $\gamma_t$ that appear in the model.
Study the *random curves* $\gamma_t$ that appear in the model.
Study the *random curves* $\gamma_t$ that appear in the model.

Use dynamic approach: parametrize $\gamma$
Study the *random curves* $\gamma_t$ that appear in the model.

Use dynamic approach: parametrize $\gamma$
- Study the *random curves* $\gamma_t$ that appear in the model.
- Use dynamic approach: parametrize $\gamma$
Study the *random curves* $\gamma_t$ that appear in the model.

Use dynamic approach: parametrize $\gamma$
Interfaces and Schramm-Loewner evolution

- Study the *random curves* $\gamma_t$ that appear in the model.
- Use dynamic approach: parametrize $\gamma$
Study the *random curves* $\gamma_t$ that appear in the model.

Understand how does the conformal map $g_t : \Omega \setminus \gamma_{[0,t]} \to \Omega$ change with $t$. 

\[ g_0 \]
Study the *random curves* $\gamma_t$ that appear in the model.

Understand how does the conformal map $g_t : \Omega \setminus \gamma_{[0,t]} \rightarrow \Omega$ change with $t$. 

\[ g_{t_1} \]

\[ a \]

\[ a_{t_1} \]
Study the *random curves* $\gamma_t$ that appear in the model.

Understand how does the conformal map $g_t : \Omega \setminus \gamma_{[0,t]} \rightarrow \Omega$ change with $t$. 

![Diagram showing random curves and conformal maps](image)
Interfaces and Schramm-Loewner evolution

- Study the *random curves* $\gamma_t$ that appear in the model.
- Understand how does the conformal map $g_t : \Omega \setminus \gamma_{[0,t]} \to \Omega$ change with $t$. 

![Diagram showing random curves and conformal maps](image)
Interfaces and Schramm-Loewner evolution

- Study the *random curves* $\gamma_t$ that appear in the model.
- Understand how does the conformal map $g_t : \Omega \setminus \gamma_{[0,t]} \to \Omega$ change with $t$. 

![Diagram of random curves and a conformal map](image)
Loewner evolution in the half-plane

For any curve, the maps $g_t$ satisfy the equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - a_t}.$$ 

The curve is completely described by the “driving force” $a_t$. 

Konstantin Izyurov (SPbSU)

SLE(3) and the Ising interfaces

June 16, 2012 6 / 28
Chordal SLE$_3$ case

S. Smirnov et al., 2007-2011: in the case of Dobrushin boundary conditions, a single interface converges to a conformally invariant process SLE$_3$. 
Chordal SLE$_3$ case

S. Smirnov et al., 2007-2011: in the case of Dobrushin boundary conditions, a single interface converges to a conformally invariant process SLE$_3$.
Ising model and SLE: known results

Chordal SLE$_3$ case
S. Smirnov et al., 2007-2011: in the case of Dobrushin boundary conditions, a single interface converges to a conformally invariant process SLE$_3$.

Dipolar SLE$_3$ case
C. Hongler, K. Kytölä: in the case of mixed “+”, “−” and free boundary conditions, the single interface converges to dipolar SLE$_3$. 
Main results: convergence of interfaces

Theorem 2 (K. I., 2011)
The multiple Ising interfaces in multiply connected domains have a conformally invariant limit. This limiting process can be described as a Loewner evolution with driving force satisfying $\frac{da}{dt} = \sqrt{3}dB_t + 3\partial a \log Zdt$, where $Z$ is the scaling limit of partition function and can be expressed via holomorphic spinor observables.
Main results: convergence of interfaces

Theorem 2 (K. I., 2011)

The multiple Ising interfaces in multiply connected domains have a conformally invariant limit. This limiting process can be described as a Loewner evolution with driving force satisfying

\[ da_t = \sqrt{3} dB_t + 3 \partial_a \log Z dt, \]

where \( Z \) is the scaling limit of partition function and can be expressed via holomorphic spinor observables.
SLE in multiply-connected domains

Take a conformal map $g_0$ to “almost upper half-plane” and use chordal Loewner evolution
SLE in multiply connected domains

\[ da_1(t) = \sqrt{3} dB(t) + 3 \partial_{a_1} \log Z(t) dt, \text{ where} \]

\[ Z(t) = Z(\Omega(t), a_1(t), \ldots, a_{2n}(t)). \]

\[ \partial_{a_1} \log Z := \lim_{\Delta \to 0} \lim_{\delta \to 0} \Delta^{-1} \left[ \frac{Z_{a_1+\Delta,a_2,\ldots,a_{2n}}}{Z_{a_1,a_2,\ldots,a_{2n}}} - 1 \right] \]
\[ da_1(t) = \sqrt{3} dB(t) + 3 \partial_{a_1} \log Z(t) dt, \quad \text{where} \]

\[ Z(t) = Z(\Omega(t), a_1(t), \ldots, a_{2n}(t)). \]

\[ \partial_{a_1} \log Z := \lim_{\Delta \to 0} \lim_{\delta \to 0} \Delta^{-1} \left[ \frac{Z_{a_1+\Delta, a_2, \ldots, a_{2n}} - 1}{Z_{a_1, a_2, \ldots, a_{2n}}} \right] \]
SLE in multiply connected domains

\[ da_1(t) = \sqrt{3} dB(t) + 3 \partial a_1 \log Z(t) dt, \text{ where} \]

\[ Z(t) = Z(\Omega(t), a_1(t), \ldots, a_{2n}(t)). \]

\[ \partial a_1 \log Z := \lim_{\Delta \to 0} \lim_{\delta \to 0} \Delta^{-1} \left[ \frac{Z_{a_1+\Delta, a_2, \ldots, a_{2n}}}{Z_{a_1, a_2, \ldots, a_{2n}}} - 1 \right] \]
SLE in multiply connected domains

\[ da_1(t) = \sqrt{3} dB(t) + 3 \partial_{a_1} \log Z(t) dt, \text{ where} \]
\[ Z(t) = Z(\Omega(t), a_1(t), \ldots, a_{2n}(t)). \]
\[ \partial_{a_1} \log Z := \lim_{\Delta \to 0} \lim_{\delta \to 0} \Delta^{-1} \left[ \frac{Z_{a_1+\Delta,a_2,\ldots,a_{2n}}}{Z_{a_1,a_2,\ldots,a_{2n}}} - 1 \right] \]
Multiple SLE’s in simply-connected domain
Multiple SLE’s in simply-connected domain

or

\[ a_4 = \infty \]

or

\[ a_4 = \infty \]
Four marked points in a simply-connected domain

If the domain is the upper half-plane $\mathbb{H}$ and four points $a', a, a'', \infty \in \overline{\mathbb{R}}$, then the driving force is $da_t = \sqrt{3}dB_t + D_t dt$

\[
D_t = 3 \left( \frac{1}{(a_t - a''_t)^2} - \frac{1}{(a_t - a'_t)^2} \right) \cdot \left( \frac{1}{a'_t - a''_t} - \frac{1}{a_t - a''_t} + \frac{1}{a_t - a'_t} \right)^{-1}
\]
Four marked points in a simply-connected domain

If the domain is the upper half-plane $\mathbb{H}$ and four points $a', a, a'', \infty \in \overline{\mathbb{R}}$, then the driving force is $da_t = \sqrt{3}dB_t + D_t dt$

$$D_t = 3 \left( \frac{1}{(a_t - a'')^2} - \frac{1}{(a_t - a')^2} \right) \cdot \left( \frac{1}{a_t' - a''} - \frac{1}{a_t - a'} + \frac{1}{a_t - a'} \right)^{-1}$$

Annulus case
Annulus SLE

SLE in the annulus $e^{-p} < |z| < 1$:

$$\partial_t g_t(z) = g_t(z)S_{a_t}^{p-t}(g_t(z))$$

$$S_a^p(z) = v.p. \sum_{k=-\infty}^{\infty} \frac{e^{2pk} + \frac{z}{a}}{e^{2pk} - \frac{z}{a}}$$

$$a_t = e^{i\theta_t}; \quad d\theta_t = \sqrt{3}dB_t + D_t dt$$

“Chordal” case, $|b_t| = 1$:

$$D_t = \text{Im} \frac{3}{2} \left[ S_{a_t}^{p-t}(b_t) + S_{a_t}^{p-t}(-\frac{a_t^2 e^{-p}}{b_t}) \right]$$

“Radial” case, $|b_t| = e^{-p+t}$:

$$D_t = \text{Im} \frac{3}{2} \left[ S_{a_t}^{p-t}(b_t) - S_{-a_t}^{p-t}(b_t) \right]$$
Radial SLE
Smirnov’s observable

Smirnov’s fermionic observable:

\[ F_{\Omega^\delta}(a, z) = \sum_{S \in \text{Conf}_{a, z}} x^{|S|} e^{-i \frac{w(\gamma:a \rightarrow z)}{2}} \]
Scaling limit of the observable

The function \( F_{\Omega \delta}(a, z) \) is discrete holomorphic; it satisfies Riemann-Hilbert boundary conditions

\[
F_{\Omega \delta}(a, z) \parallel 1^{\sqrt{\nu}z}, \quad z \in \partial \Omega \setminus \{a\}
\]

This allows one to prove that after a proper normalization, as \( \Omega \delta \to \Omega \),

\[
F_{\Omega \delta}(a, z) \to f_{\Omega}(a, z)
\]

where \( f_{\Omega}(a, z) \) is holomorphic

\[
f_{\Omega}(a, z) := \sqrt{i^{\nu}a z - a + O(1)}, \quad z \to a
\]
The function $F_{\Omega^\delta}(a, z)$ is *discrete holomorphic*;
Scaling limit of the observable

- The function $F_{\Omega^\delta}(a, z)$ is discrete holomorphic;
- It satisfies Riemann-Hilbert boundary conditions

$$F_{\Omega^\delta}(a, z) \parallel \frac{1}{\sqrt{\nu_z}}, \quad z \in \partial\Omega^\delta \setminus \{a\}.$$
The function $F_{\Omega^\delta}(a, z)$ is *discrete holomorphic*;

It satisfies Riemann-Hilbert boundary conditions

$$F_{\Omega^\delta}(a, z) \parallel \frac{1}{\sqrt{\nu(z)}}, \quad z \in \partial\Omega^\delta \setminus \{a\}.$$ 

This allows one to prove that after a proper normalization, as $\Omega^\delta \to \Omega$,

$$F_{\Omega^\delta}(a, z) \to f_{\Omega}(a, z),$$

where

$$f_{\Omega}(a, z) \quad \text{is holomorphic}$$

$$f_{\Omega}(a, z) \parallel \frac{1}{\sqrt{\nu(z)}}, \quad z \in \partial\Omega \setminus \{a\}$$

$$f(a, z) := \frac{\sqrt{i\nu a}}{z-a} + O(1), \quad z \to a$$
Conformal covariance

\[ f_\Omega(a, z) \text{ is holomorphic} \]
\[ f_\Omega(a, z) \parallel \frac{1}{\sqrt{i\nu z}}, \quad z \in \partial\Omega \setminus \{a\} \]
\[ f(a, z) := \frac{\sqrt{i\nu a}}{z-a} + O(1), \quad z \to a \]

The function with such properties is unique up to a multiplicative constant and is conformally covariant:

\[ f_\Omega(a, z) = (\phi'(z))^{1/2} f_\phi(\Omega)(\phi(a), \phi(z)) \]

and in fact

\[ f_{\mathbb{C}_+}(a, z) = \frac{1}{z-a} \]
Let $\gamma_n := e_1 \cup e_2 \cdots \cup e_n$ be the initial segment of the interface. Then

$$
\frac{F_{\Omega^\delta \setminus \gamma_n}(z)}{F_{\Omega^\delta \setminus \gamma_n}(b)}
$$

is a martingale w. r. t. the filtration $\{\sigma(\gamma_n)\}$. 
Proof:

\[ F_{\Omega^\delta}(a, b) = \sum_{S \in \text{Conf}_{a,b}} x|S|e^{-i\frac{w(\gamma:a\rightarrow b)}{2}} = e^{-i\frac{w(\gamma:a\rightarrow b)}{2}}Z_{\Omega,a,b}. \]

\[ p_j := \mathbb{P}(\gamma_1 = a_j) = \frac{xZ_{\Omega\setminus a_j,a_j,b}}{Z_{\Omega,a,b}}, \quad j = 1, 2, 3. \]

Thus:

\[ \mathbb{E} \left[ \frac{F_{\Omega^\delta \setminus \gamma_1}(\gamma_1, z)}{F_{\Omega^\delta \setminus \gamma_1}(\gamma_1, b)} \right] = \sum_{j=1,2,3} p_j \frac{F_{\Omega^\delta \setminus a_j}(z)}{F_{\Omega^\delta \setminus a_j}(b)} = \]

\[ = \frac{\sum_{j=1,2,3} xF_{\Omega^\delta \setminus a_j}(z)}{e^{-i\frac{w(\gamma:a\rightarrow b)}{2}}Z_{\Omega,a,b}} = \frac{F_{\Omega^\delta}(a, z)}{F_{\Omega^\delta}(a, b)}. \]
Martingale property fails for radial-type configurations

\[ F_{\Omega^\delta}(a, b) = \sum_{S \in \text{Conf}_{a,b}} x |S| e^{-i \frac{w(\gamma:a\rightarrow b)}{2}} \]
Martingale property fails for radial-type configurations

\[ F_{\Omega^\delta}(a, b) = \sum_{S \in \text{Conf}_{a,b}} x^{|S|} e^{-i \frac{w(\gamma:a\rightarrow b)}{2}} \]
The remedy: spinor observables

\[ F_1(a, z) = \sum_{S \in \text{Conf}_{a, z}} \chi(S) e^{-iw(\gamma:a \to b)/2} (-1)^{l(s)+1(\gamma:a \to z)}, \]

where \( l(e, z) \) is the number of non-trivial loops and \( 1(\gamma : a \to z) \) is the indicator of the event that \( \gamma \) arrives at the correct sheet.
Martingale property is regained

\[ F_1(a, b) = \sum_{S \in \text{Conf}_{a,b}} x^{|S|} e^{-i \frac{w(\gamma:a\rightarrow b)}{2}} (-1)^1(\gamma:a\rightarrow b) \]
Martingale property is regained

\[ F_1(a, b) = \sum_{S \in \text{Conf}_{a,b}} x^{|S|} e^{-i \frac{w(\gamma:a\rightarrow b)}{2}} (-1)^{1(\gamma:a\rightarrow b)} \]
Multiple marked points

(A)

(B)
How to extract the driving force?

Consider the simply-connected domain with just two marked points: if \( f \) is the scaling limit of the observable \( F \), one has explicitly

\[
f(a, z) := f_{\mathbb{C}^+}(a, z) = \frac{1}{z - a}
\]
How to extract the driving force?

Consider the simply-connected domain with just two marked points: if $f$ is the scaling limit of the observable $F$, one has explicitly

$$f(a, z) := f_{\mathbb{C}^+}(a, z) = \frac{1}{z - a}$$

Use the conformal covariance

$$M_t := \frac{f_{\mathbb{C}^+ \setminus \gamma_t}(a, z)}{f_{\mathbb{C}^+ \setminus \gamma_t}(a, b)} = \frac{f_{\mathbb{C}^+}(a_t, z_t)}{f_{\mathbb{C}^+}(a_t, b_t)} \left( \frac{g'_t(z)}{g'_t(b)} \right)^{1/2}$$

and assume that the driving process has a form $da_t = A_t dB_t + D_t dt$ then Itô calculus shows
\[dM_t = \left[ A_t^2 \partial_{aa} f(a_t, z_t) + \frac{\partial_z f(a_t, z_t)}{z_t - a_t} - \frac{f(a_t, z_t)}{(z_t - a_t)^2} + \\
D_t \partial_a f(a_t, z_t) - \frac{A_t^2 \partial_a f(a_t, z_t) \partial_a f(a_t, b_t)}{f(a_t, b_t)} \\
+ (...) f(a_t, z_t)] (...) dt + [...] dB_t.\]
\[ dM_t = \left[ A_t^2 \partial_{aa} f(a_t, z_t) + \frac{\partial_z f(a_t, z_t)}{z_t - a_t} - \frac{f(a_t, z_t)}{(z_t - a_t)^2} + 
\right. \\
\left. D_t \partial_a f(a_t, z_t) - \frac{A_t^2 \partial_a f(a_t, z_t) \partial_a f(a_t, b_t)}{f(a_t, b_t)} \right. \\
\left. + (\ldots) f(a_t, z_t) \right] (\ldots) dt + \ldots dB_t. \]

Substituting \( f(a_t, z_t) = \frac{1}{z-a} \) yields

\[ dM_t = \\
\left[ \frac{(A_t^2 - 3)}{(z_t - a_t)^3} + \frac{D_t - A_t^2 \partial_a \log f(a_t, b_t)}{(z_t - a_t)^2} + \frac{(...)}{z_t - a_t} \right] (\ldots) dt + \ldots dB_t. \]
\[ dM_t = \left[ A_t^2 \partial_a f(a_t, z_t) + \frac{\partial_z f(a_t, z_t)}{z_t - a_t} - \frac{f(a_t, z_t)}{(z_t - a_t)^2} + \right. \]
\[ D_t \partial_a f(a_t, z_t) - \frac{A_t^2 \partial_a f(a_t, z_t) \partial_a f(a_t, b_t)}{f(a_t, b_t)} \]
\[ \left. + \left( \ldots f(a_t, z_t) \right) (\ldots) dt + [\ldots] dB_t. \right] \]

More generally, \( f(a_t, z_t) = \frac{1}{z - a} + o(1), \; z \to a; \) and

\[ dM_t = \left[ \frac{(A_t^2 - 3)}{(z_t - a_t)^3} + \frac{D_t - A_t^2 \partial_a \log f(a_t, b_t)}{(z_t - a_t)^2} + O\left( \frac{1}{z_t - a_t} \right) \right] (\ldots) dt + [\ldots] dB_t. \]
The driving process is expressed through the observables \( f_\Omega(a, \cdot) \), which is the (unique) solution to the following boundary value problem:

\[
\begin{align*}
    f_\Omega(a, z) & \quad \text{is holomorphic} \\
    f_\Omega(a, z) \parallel \frac{1}{\sqrt{i \nu z}}, & \quad z \in \partial \Omega \setminus \{a\} \\
    f_\Omega(a, z) & := \frac{\sqrt{i \nu a}}{z-a} + O(1), \quad z \to a,
\end{align*}
\]

and \( f_\Omega(a, z) \) is a spinor w. r. t. a suitable double cover of the domain.
The driving process is expressed through the observables $f_\Omega(a, \cdot)$, which is the (unique) solution to the following boundary value problem:

- $f_\Omega(a, z)$ is holomorphic
- $f_\Omega(a, z) \parallel \frac{1}{\sqrt{ivz}}, \quad z \in \partial \Omega \setminus \{a\}$
- $f_\Omega(a, z) := \frac{\sqrt{iv_a}}{z-a} + O(1), \quad z \to a,$

and $f_\Omega(a, z)$ is a spinor w. r. t. a suitable double cover of the domain.

- **Uniqueness:** easy;
The driving process is expressed through the observables $f_\Omega(a, \cdot)$, which is the (unique) solution to the following boundary value problem:

$$f_\Omega(a, z) \text{ is holomorphic}$$

$$f_\Omega(a, z) \parallel \frac{1}{\sqrt{i\nu z}}, \quad z \in \partial \Omega \setminus \{a\}$$

$$f_\Omega(a, z) := \frac{\sqrt{i\nu a}}{z-a} + O(1), \quad z \to a,$$

and $f_\Omega(a, z)$ is a spinor w. r. t. a suitable double cover of the domain.

- **Uniqueness:** easy;
- **Existence:** by convergence of the discrete observables; see also C. Hongler and D. H. Phong (2012).