Where? – Simons Center, Stony Brook
When? – January–April 2013
Organizers: Ilia Binder, John Cardy, Andrei Okounkov, Paul Wiegmann

Workshops:
- “Integrable structures in random processes" January 21–25 (Binder-Okounkov)
- “Random Tiling" February 11–15 (Kenyon-de Gier-Nienhuis)
- “Conformal invariance in continuous and discrete systems" April 8–12 (Cardy-Wiegmann-Lawler)

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Holomorphic Parafermions on the Lattice and in Conformal Field Theory

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Outline

- some pre-history
- discretely holomorphic parafermions in Ising and $\mathbb{Z}_N$ models
- discretely holomorphic parafermions from random curves
- discrete holomorphicity and integrability
- holomorphic parafermions in CFT and SLE
Green [1953] invented the term parastatistics to describe multi-particle states in quantum field theory which transform according to non-trivial representations of $S_n$.

Fradkin-Kadanoff [1980] identified parafermions in classical 2d lattice models which were generalisations of Kaufman’s 1949 Ising model fermions: they argued that in the scaling limit at criticality

$$\langle \psi_\sigma(z_1)\psi_\sigma(z_2) \rangle \propto (z_1 - z_2)^{-2\Delta}(z_1^* - z_2^*)^{-2\overline{\Delta}}$$

$\sigma = \Delta - \overline{\Delta}$ is ‘fractional spin’

if $\Delta$ or $\overline{\Delta} = 0$, $\psi_\sigma$ is (anti-)holomorphic

in 2d quantum models (eg for the fractional quantum Hall effect) these were called anyons [Wilczek, 1982]
Example: the Ising model

- $\mathbb{Z}^2$ lattice, degrees of freedom $s_r = \pm 1$, weights

$$e^{\sum_{rr'} J_{rr'} s(r)s(r')} \propto \prod_{rr'} \left( 1 + x_{rr'} s(r)s(r') \right)$$

- $\mathbb{Z}_2$ symmetry under $s(r) \rightarrow -s(r)$
- Disorder variable $\mu(R)$: $s(r)$ identified with $-s(r)$ as $r$ goes around $R$
- Equivalent to taking $x_{rr'} \rightarrow -x_{rr'}$ on edges $(rr')$ which cross a ‘string’ attached to $R$: 

![Diagram of a lattice with a string and a dot indicating the string's attachment point]
define fermion $\psi_\sigma(rR)$ on the edge $(rR)$:

$$\psi_\sigma(rR) = s(r) \cdot \mu(R) e^{-i\sigma \theta_{rR}}$$

$$\mu(R_4) = \frac{1 - xs(r_1)s(r_2)}{1 + xs(r_1)s(r_2)} \mu(R_3)$$
\[
(1 + xs(r_1)s(r_2)) \mu(R_4) = (1 - xs(r_1)s(r_2)) \mu(R_3)
\]

- multiply both sides by \(s(r_1)\) and \(s(r_2)\) and use \(s^2 = 1\):

\[
s(r_1)\mu(R_4) + xs(r_2)\mu(R_4) = s(r_1)\mu(R_3) - xs(r_2)\mu(R_3)
\]

\[
xs(r_1)\mu(R_4) + s(r_2)\mu(R_4) = -xs(r_1)\mu(R_3) + s(r_2)\mu(R_3)
\]

- on the other hand we have discrete holomorphicity if

\[
e^{i\pi/4}\psi_{13} + e^{3i\pi/4}\psi_{23} + e^{5i\pi/4}\psi_{24} + e^{7i\pi/4}\psi_{14} = 0
\]

- these are consistent if \(\sigma = \frac{1}{2}\) and \(x = \tanh J = \sqrt{2} - 1\) - the condition for the isotropic Ising model to be critical!
\[
(1 + xs(r_1)s(r_2)) \mu(R_4) = (1 - xs(r_1)s(r_2)) \mu(R_3)
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- this is true inside all correlation functions with other observables elsewhere and with all (compatible) boundary conditions

Holomorphic Parafermions
more generally we can take different couplings $J, J'$ in different directions

we then get discrete holomorphicity only if we embed the lattice in $\mathbb{R}^2$ so each face is a rhombus of interior angle $\alpha$ and

$$\tanh J = \tan \left( \frac{\pi - \alpha}{4} \right), \quad \tanh J' = \tan \left( \frac{\alpha}{4} \right)$$

which implies $\sinh 2J \sinh 2J' = 1$, the condition for criticality
more generally we can take \( s(r) \in (1, \omega, \ldots, \omega^{N-1}) \) where 
\[
\omega = e^{2\pi i/N}
\]

nearest neighbour interaction with most general weights

\[
\prod_{rr'} \left( 1 + \sum_{j=1}^{N-1} w_j (s(r)^* s(r'))^j + \text{c.c.} \right)
\]

and similarly \( w'_j \) on the vertical edges

define disorder variables \( \mu(R) \) by \( s \rightarrow \omega s \) across the string

define parafermions

\[
\psi_\sigma \equiv e^{-i\sigma \theta^r} s(r) \mu(R)
\]
these are discretely holomorphic [Rajabpour, JC 2008] if \( \sigma = (N - 1)/N \) and

\[
    w_j = x_j(\alpha) = \prod_{j'=0}^{j-1} \frac{\sin(2\pi j' + \alpha)/2N}{\sin(2\pi (j' + 1) - \alpha)/2N),
    w'_j = x_j(\pi - \alpha)
\]

these are the weights of the Fateev-Zamolodchikov model [1982], which is critical and integrable in the sense of Yang-Baxter

F-Z [1985] assumed these parafermions are holomorphic in the scaling limit and used them as building blocks of the corresponding CFT.
consider a planar lattice built out of elementary faces or plaquettes, e.g. $\mathbb{Z}^2$

the degrees of freedom $(a, b, \cdots)$ live on the vertices and the weight for a given configuration is

$$\propto \prod_{\text{faces}} W(a, b, c, d)$$

such models can also be realised as vertex models with degrees of freedom on the edges
the weights $W(a, b, c, d; u)$ depend on a real variable $u$ (spectral parameter) such that, when summed over the central degree of freedom $c$

$$
\sum_c W(., ., c, .; u) W(., ., ., v) W(., ., c; u - v)
$$

$$
= \sum_c W(., c, ., u - v) W(., ., c; v) W(., c, ., ; u)
$$

for Ising and $\mathbb{Z}_N$ models $W(a, b, c, d; u) \propto w(a, c; u)w(b, d; u')$

and Y-B equations are usually called star-triangle relations
the discretely holomorphic weights of the Ising and $Z_N$ models satisfy the Yang-Baxter equations with $\alpha$ identified as the spectral parameter $u$

the spectral parameter tells us how to embed the lattice in $\mathbb{R}^2$ so as to get discrete holomorphicity (and conjecturally full conformal invariance in the scaling limit)
More general lattices

- this extends to an inhomogeneous 2-colourable ‘Baxter lattice’: such a lattice can always be embedded in $\mathbb{R}^2$ so that all its faces are rhombi, so it is isoradial: if the local weights are those determined by the local angle $\alpha$ then $\psi_\sigma$ is discretely holomorphic.
the Yang-Baxter equations are equivalent to saying that different tilings of a hexagon do not change measure in exterior
many (but not all?) interesting lattice models may also be realised in terms of non-intersecting curves

Smirnov showed how to construct discretely holomorphic observables for some simple examples

a more instructive example is the $O(n)$ model on $\mathbb{Z}^2$ lattice [Nienhuis,Blöte]
in this case one considers curves $\gamma_{z_0z_e}$ from some point $z_0$ ending at a given edge $z_e$, with turning angle $\theta_{z_0z_e}$

- $\psi_\sigma(z_e) \equiv \mathbb{E}[e^{-i\sigma z_0z_e}]$

- $\sum_{\square} \psi_\sigma(z_e) \delta z_e = 0$ if the weights satisfy the critical Y-B equations and the faces are embedded in $\mathbb{R}^2$ as rhombi with angle $\alpha = \theta$ [Ikhlef, JC 2009]
Yang-Baxter equations are cubic functional equations for $W(\cdots; u)$

\[
\begin{align*}
W(u, u-v, v) &= W(u-v, u, v)
\end{align*}
\]

Discrete holomorphicity is a linear condition on $W(\cdots; u)$ for a fixed $u$.
Yang-Baxter equations are cubic functional equations for $W(\cdots; u)$

\[
\begin{align*}
u & = u - v \\
v & = u - v
\end{align*}
\]

discrete holomorphicity is a linear condition on $W(\cdots; u)$ for a fixed $u$

\[
\begin{align*}
4 & \quad 3 \\
1 & \quad 2
\end{align*}
\]

does one imply the other in general? Are there counter-examples?

connection is clearer in the limit $u - v \to 0$ when rhombus degenerates and both sets of equations simplify
Holomorphic parafermions from the continuum

- curve from $z_0$ to point on boundary of disc $z_e = z + \epsilon e^{i\theta}$
- in CFT language consider [cf. Werness’ SLE approach]

\[
\psi_\sigma(z) = \lim_{\epsilon \to 0} \int d\theta e^{-i\sigma \theta} \phi_{\text{boundary}}(z + \epsilon e^{i\theta})
\]

- [Simmons + JC] for $\sigma = \Delta_{21} = (6 - \kappa) / 2\kappa$ limit exists and $\psi_\sigma(z)$ is holomorphic
- moreover its correlators satisfy 2nd order linear differential equations wrt $z$
Holomorphic parafermions and CFT

- In the half-plane the parafermionic observable corresponds to the CFT correlation function

\[ \left\langle \phi_{21}(z_0)\psi_\sigma(z) \right\rangle_\mathbb{H} \sim (z_0 - z)^{-\Delta_{21} - \sigma} \]

- As \( z \to \text{boundary} \), \( \psi_\sigma \to \phi_{21} \)

- Conformal invariance implies that these have the same scaling dimension and hence \( \sigma = \Delta_{21} = (6 - \kappa) / 2\kappa \)
extension to $N$ curves

...this suggests that bulk holomorphic parafermions exist with 
\[ \sigma = \Delta_{N+1,1} \]

$N = 2$ is already identified in terms of boundary curves of F-K clusters [Smirnov, Riva+JC]

in CFT every boundary correlation function has a an extension into the complex $z$-plane – this suggests that to each boundary operator in a given CFT with scaling dimension $\Delta$, there exists a holomorphic bulk operator with conformal spin $\Delta$

if so, there may be a lot more possible discretely holomorphic observables out there!
Some outstanding problems

- is it always true that [cf Fendley’s talk]

  \[ \text{discrete holomorphicity } \Leftrightarrow \text{criticality} + \text{Yang-Baxter integrability?} \]

  - can one do something useful with this? [eg Smirnov–Duminil-Copin]
  - boundary extensions [see Guttmann, Ikhlef talks]

- can convergence of \( \psi_\sigma \) to a truly holomorphic quantity be proved for cases other than the Ising model?
- major problem: not enough equations!
  - can this be used to prove convergence of the measure on curves to SLE à la Smirnov?
  - can this be used as a constructive route to the full scaling theory/CFT? [cf. talks by Hongler, Chelkak, Izyurov]

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