A detour into wave function collapses
(An exercise in probability theory with a bit of quantum mechanics)

An interlude: no holomorphicity, no integrability, but discrete and a few probabilities....

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Motivations:

Experiments on QED in cavities:

*Indirect evaluations of probability distribution functions (p.d.f.) of the photon number in a cavity.*

The estimated photon number p.d.f. is recalculated after each indirect measurement via Bayesian rules arising from Q.M.:

Collapse of the p.d.f. !??

(with a realization dependent target)

«An exercise in probability theory»
Plan:

- A view on random walks.
- A quantum version of them.
- Repeated (quantum) measurements and Bayes’ law.
  (hidden random walks).
- Repeated (quantum) measurements and collapses.
  (via martingale convergence theorem).
- Pointer states.
- Time continuous limit and Brownian motions.

Classical probability theory with a bit of quantum mechanics.
Random walks... and repeated interactions.

Random walks via iterated kicks and measure

Events: \((\mathbf{\cdots})\) or \(\omega = (+, +, -, +, \cdots) = (\epsilon_1, \epsilon_2, \epsilon_2, \cdots)\)

Filtration:
\[
B_{\epsilon_1, \epsilon_2, \cdots, \epsilon_n} := \{\omega = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \text{any thing else})\}
\]
\[
\mathcal{F}_n := \sigma - \text{algebra generated by all } B_{\epsilon_1, \epsilon_2, \cdots, \epsilon_n}
\]

Gain of information as \(n\) increases.

Or... we can give ourselves the algebras \(\mathcal{B}_n\) of \(\mathcal{F}_n\) measurable functions

Quantization: Everything becomes non-commutative.....
Quantum repeated interactions:

\( P := \) Probes
\( S := \) Quantum System

\[ \mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \otimes \cdots \]

\[ \mathcal{B}_n := \mathcal{A}_s \otimes \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \otimes \mathbb{I} \]

Filtration:
\( \mathcal{A}_s \subset \mathcal{B}_n \subset \mathcal{B}_m, \quad \text{for } n < m \)

Measurements on the probe after interaction with the Q-system

Hilbert space:

Algebras of observables:

Gain of informations: Test output probe observables on the \( n \)-th first probes, but probabilistic gain because of Q.M.
Basics of quantum mechanics:

* **Kinematics**: State of a system = a vector (up to a phase) in a Hilbert space $\mathcal{H}$.

* **Dynamics**: Some linear dynamical equation for the system state (Schrodinger equation)

* **Observations**: Observables are represented by hermitian operators on $\mathcal{H}$.
  Possible outputs of measurements of the observable $O$ is an element of $\text{Spec}(O)$.

For a system in state $\mathcal{U}$, normalized $\langle \mathcal{U} | \mathcal{U} \rangle = 1$,
the probability to measure value $i$ in $\text{Spec}(O)$ is $\langle \mathcal{U} | \mathcal{P}_i \mathcal{U} \rangle$.

* **Collapse of the wave function**: After a measurement with output $i$ in $\text{Spec}(O)$,
  the system state is re-initialized to $\mathcal{P}_i \mathcal{U}$ (to be normalized)
Today’s Aim:

Pick a basis $a$ of states of the Q-system.
Reconstruct the probability distribution for the system to be in state $a$

$$Q_0(\alpha), \quad \sum_\alpha Q_0(\alpha) = 1,$$

$S :=$ Quantum System

$\mathcal{P} :=$ Probes

Measure some observables on the ($n$-th) output probes:
(not on the Q-system)

Measurements !!!

outputs: $i$ in $\mathcal{I}$

The quantum filtration is reduced to a classical filtration.

Possible outputs of the measurements: $i$ in some set $\mathcal{I}$.

After $n$ measurements: gain in information $(i_1, i_2, \cdots, i_n)$

Events are the infinite sequences of measurement outputs.

$\mathcal{F}_n := \sigma-$ algebra generated by all $B_{i_1, i_2, \cdots, i_n} \equiv \{\omega = (i_1, i_2, \cdots, i_n, \text{any thing else})\}$
QED in cavity experiments:

Iterated non demolition measurements.

System (S) = photons in a cavity.
Probes (P) = Rydberg atoms (two state systems : + or -)

Probe measurements give values + or -;
Recursion for the photon number p.d.f. from data of sequences
\[ \omega = (+, +, -, +, \cdots) = (\epsilon_1, \epsilon_2, \epsilon_2, \cdots) \]
Evolution of the probability distributions:

Quantum mechanics implies «classical» Bayes’ rules

Pick a basis $a$ of state of the Q-system.

Start with a probability distribution (initial system state):

$$Q_0(\alpha), \quad \sum_{\alpha} Q_0(\alpha) = 1,$$

Let $i$ be the output measurements on the probe.

Data (probe-system interaction) are probabilities to measure

$i$ conditioned on the Q-system to in state $a$.

$$p(i|\alpha), \quad \sum_i p(i|\alpha) = 1$$

At each step «after each cycle of system-probe interaction + probe measurement», the system probability distribution is updated.

Let $Q_{n-1}(a)$ be the probability distribution of the Q-system after $(n-1)$ cycles,

The output of the $n$-th probe measurement is $i_n$ with proba:

$$\pi_n(i_n) := \sum_{\alpha} p(i_n|\alpha) Q_{n-1}(\alpha)$$

Then,

$$Q_n(\alpha) = \frac{1}{Z_n} p(i_n|\alpha) Q_{n-1}(\alpha), \quad \text{with} \quad Z_n = \sum_{\alpha} p(i_n|\alpha) Q_{n-1}(\alpha)$$
Bayes’ rule from Quantum Mechanics:

* For probabilists: ... just admit...

* For physicists: ... just quantum mechanics...

The only delicate point: we suppose that there is a basis of state $\alpha$ in $\mathcal{H}_s$ preserved by the probe-system interaction, i.e.:

$$U |\alpha\rangle \otimes |\phi\rangle = |\alpha\rangle \otimes U_\alpha |\phi\rangle$$

for the $U$ the evolution operator during the probe-system interaction

Gain of information because probe-system entanglement.
Collapse of the p.d.f. \( Q_n(\alpha) \) as \( n \to \infty \)

\[
Q_n(\alpha) = \frac{1}{Z_n} p(i_n|\alpha) Q_{n-1}(\alpha), \quad \text{with} \quad Z_n = \sum_{\alpha} p(i_n|\alpha) Q_{n-1}(\alpha)
\]

* Peaked distributions are stable (stability of the pointer states):

\[
Q_n(\alpha) = \delta_{\alpha;\gamma} \quad \text{are solutions.}
\]

Then, output \( i_n \) with probability: \( p(i_n|\gamma) \)

* Probability distributions converge a.s. towards peaked distributions (collapse of the wave function):

\[
\lim_{n \to \infty} Q_n(\alpha) = \delta_{\alpha,\gamma} \quad \text{with a realization dependent target} \quad \gamma
\]

\[
\text{Prob}[\gamma = \beta] = Q_0(\beta) \quad \text{(Von Neumann rules for quantum measurements)}
\]

* The convergence is exponential: \( Q_n(\alpha) \sim \exp[-n S(\gamma|\alpha)] \) \((\alpha \neq \gamma)\)

with \( S(\gamma|\alpha) = -\sum_i p(i|\gamma) \log \frac{p(i|\alpha)}{p(i|\gamma)} \) a relative entropy.

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Proof: \( Q_n \) are bounded martingales.

\[ Q_n(\alpha) = \frac{1}{Z_n} p(i_n|\alpha) Q_{n-1}(\alpha), \quad \text{with} \quad Z_n = \sum_{\alpha} p(i_n|\alpha) Q_{n-1}(\alpha) \]

with probability \( \pi_n(i_n) := \sum_{\alpha} p(i_n|\alpha) Q_{n-1}(\alpha) \), thus:

\[ \mathbb{E}[Q_n(\alpha)|\mathcal{F}_{n-1}] = \sum_i \pi_n(i) \frac{p(i|\alpha)Q_{n-1}(\alpha)}{Z_n} = Q_{n-1}(\alpha) \]

* By the martingale convergence th., \( Q_n \) converges a.s. and in \( L^1 \):

\[ Q_\infty(\alpha) = \lim_{n \to \infty} Q_n(\alpha) \quad \text{exists} \]

* If all distributions \( p(i|a) \) are different (for \( a, a' \) diff.),
(by the stationary point of the recursion relation)

By the martingale property,

\[ \mathbb{E}[Q_\infty(\beta)] = Q_0(\beta) \quad \Rightarrow \quad \text{Prob}[\gamma_\omega = \beta] = Q_0(\beta) \]

* At large \( n \), each \( i \) occurs with probability \( p(i|\gamma_\omega) \) frequencies:

\[ N_n(i) \overset{\text{as} n \to \infty}{\sim} np(i|\gamma_\omega) \]

\[ Q_n(\alpha) = Q_0(\alpha) \frac{\prod_i p(i|\alpha)^{N_n(i|\alpha)}}{\sum_{\beta} Q_0(\beta) \prod_i p(i|\beta)^{N_n(i|\beta)}} \approx \frac{Q_0(\alpha)}{Q_0(\gamma_\omega)} \exp[-nS(\gamma_\omega|\alpha)] \]

only one term is dominant in the sum \( Z_n \).

* Independence w.r.t to the initial trial distribution (not \( Q_0 \), because the latter is a priori unknown).
Macroscopic measurement apparatus: Measure whether the system is in state $a$, i.e. measure observable with eigenstates $a$.

Data of the apparatus: the p.d.f.‘s $p(i|\alpha)$ on $\mathcal{I}$, for all $\alpha$.

$S := \text{Quantum System}$

For each infinite cycle, the apparatus provide the infinite sequence, $(i_1, \cdots, i_n, \cdots)$

**Reader:** Compare the empirical histogram of the output measurements, with the given distributions $p(i|a)$

Target state = result of the measure

Generalization: with different probes, probe measurements, etc. randomly chosen.
Time continuous limit:

If $T$ the time duration of the probe-system interaction, rescale the interaction hamiltonian by $T^{1/2}$.

\[ p(i|\alpha) = p_0(i) [1 + \sqrt{T} \Gamma(i|\alpha) + \cdots] \]

* Behavior of the counting processes (frequencies $N_n(i)$) via a Doob decomposition:

\[ N_n(i) = X_n(i) + \sum_{k=1}^{n} \pi_k(i), \quad \text{with} \quad \pi_k(i) = \sum_{\alpha} p(i|\alpha) Q_{k-1}(\alpha) \]

A $\mathcal{F}_n$ martingale

An increasing predictable process

Proba to find $i$ in the $k$-th probe measurement

* In the continuous limit, $X_n(i)$ converge towards Brownian motions:

\[ dX_t(i) dX_t(j) = (p_0(i) \delta_{i,j} - p_0(i)p_0(j)) \, dt \]

* The p.d.f. $Q_t(\alpha)$ converge to solutions of the (non-linear) stochastic equation:

\[ dQ_t(\alpha) = Q_t(\alpha) \sum_i (\Gamma(i|\alpha) - \langle \Gamma(i|\alpha) \rangle_t) \, dX_t(i) \quad \text{with} \quad \langle \Gamma(i|\alpha) \rangle_t = \sum_{\beta} \Gamma(i|\beta) Q_t(\beta) \]

* Analysis, similarly as in the discrete case, can be done.

Differential eq. for the density matrix can be found, so-called Belavkin eqs., which describe «time-continuous measurements». 
Understanding the collapses of these probability distribution functions: An interesting exercise in probability theory, .... with some physical applications.

Thank you.