On Tuesday, November 16, the classes start at 12:30.

1. Let $(M,g)$ and $(\tilde{M},\tilde{g})$ be Riemannian manifolds and $\varphi: M \to \tilde{M}$ an isometry. Prove that the curvature tensor fields $R$ of $M$ and $\tilde{R}$ of $\tilde{M}$ satisfy an equation
   \[ \tilde{R}(\varphi_*X, \varphi_*Y)\varphi_*Z = \varphi_*(R(X,Y)Z) \]
   for all $X,Y,Z \in \mathcal{T}(M)$. Prove furthermore that the Riemannian curvature tensors $R$ and $\tilde{R}$ satisfy $\varphi^*\tilde{R} = R$.
   You may use the fact that $\varphi^*\tilde{\nabla} = \nabla$, that is,
   \[ \varphi_*(\nabla_XY) = \tilde{\nabla}_{\varphi_*X}(\varphi_*Y). \]

2. Let $M$ be a Riemannian manifold and $\gamma: [0,b] \to M$ a $C^\infty$-path. Let $V$ be a smooth ($C^\infty$) vector field along $\gamma$ such that $V_0 = 0$. Prove that
   \[ (R(V,\gamma')\gamma)'_0 = (R(V',\gamma')\gamma)_0. \]

3. Complete the proof of Theorem 3.3 (Chapter VI in the handwritten lecture notes): Let $\gamma: [a,b] \to M$ be a geodesic. If $p = \gamma_a$ is not conjugate to $q = \gamma_b$ and $v_1 \in T_pM$, $v_2 \in T_qM$, then there exists a unique Jacobi field $V$ along $\gamma$ such that $V_a = v_1$ and $V_b = v_2$.
   (Uniqueness is proven in the lecture notes.)

4. Let $M$ be a Riemannian manifold with sectional curvature identically zero. Show that, for every $p \in M$, the mapping $\exp_p: B(0,\varepsilon) \to B(p,\varepsilon)$ is an isometry whenever $B(p,\varepsilon)$ is a normal ball at $p$.

5. Let $M$ be a Riemannian manifold, $p \in M$, $x,y,z \in T_pM$, $|x| = 1$ and $\gamma = \gamma^x$. Let $Y$ and $Z$ be Jacobi fields along $\gamma$ such that $Y_0 = 0$, $Y'_0 = (D_tY)_0 = y$, $Z_0 = 0$, and $Z'_0 = (D_tZ)_0 = z$. Prove that
   \[ \langle Y_t, Z_t \rangle = t^2 \langle y, z \rangle - \frac{t^4}{3} \langle R(y,x)x, z \rangle + O(t^5). \]