1. Prove the following version of the Gauss lemma: Let \( p \in M \) and \( v \in T_pM \) a vector such that \( \exp_p v \) is defined. Let \( w \in T_v(T_pM) = T_pM \). Then
\[
\langle \exp_{p*}v(v), \exp_{p*}v(w) \rangle = \langle v, w \rangle.
\]

2. Show that any connected Riemannian manifold \((M, g)\) admits a Riemannian metric \( \tilde{g} = \varphi g \), where \( \varphi : M \to \mathbb{R} \) is a positive \( C^\infty \)-function, such that \((M, \tilde{g})\) is bounded. In other words, there exists a constant \( C \) such that \( d_{\tilde{g}}(x,y) \leq C \) for all \( x, y \in M \).

[Hint: The following facts may be useful. (a): If \( h : M \to \mathbb{R} \) is a non-negative continuous function, then there exists a \( C^\infty \)-function \( f : M \to \mathbb{R} \) s.t. \( f(x) > h(x) \) for all \( x \in M \). (b): For every \( \varepsilon > 0 \) and for every \( p, q \in M \), there exists an admissible path \( \gamma : [0, L] \to M \) such that \( L = \ell(\gamma) \leq d(p, q) + \varepsilon \) and \( |\dot{\gamma}_t| = 1 \) except for finitely many \( t \in [0, L] \).]

3. Let \( M \) and \( N \) be Riemannian manifolds and \( f : M \to N \) a diffeomorphism. Suppose that \( N \) is complete and that there exists a constant \( c > 0 \) such that
\[
|v| \geq c|f_*p v|
\]
for all \( p \in M \) and for all \( v \in T_pM \). Prove that \( M \) is complete.

4. (Corrected version) Let \( M \) be a complete connected Riemannian manifold, \( N \) a Riemannian manifold and \( f : M \to N \) a smooth mapping that is a local isometry. Suppose that for every \( x, y \in N \) there exists a unique geodesic from \( x \) to \( y \). Prove that \( f \) is bijective (and hence an isometry).

[You may use the fact that local isometries preserve geodesics.]

5. Let \( M \) be a smooth manifold, \( N \) a Riemannian manifold and \( f : M \to N \) a surjective local diffeomorphism. Introduce on \( M \) a Riemannian metric such that \( f \) is a local isometry. Furthermore, show by examples that \( M \) (equipped with the Riemannian metric introduced above) need not be complete even if \( N \) is complete.