1. Prove Lemma 5.2 (Chapter IV in the lecture notes): Let $\gamma : [a, b] \to M$ be admissible and $V$ a vector field along $\gamma$. Then there exists $\Gamma$, a variation of $\gamma$, such that $V$ is the variation field of $\Gamma$. If $V$ is proper, $\Gamma$ can be taken to be proper as well.

2. Let $B(p, r) = \exp_p B(0, r)$ be a normal ball such that $\partial B(p, r)$ is a normal sphere ($= \exp_p \partial B(0, r)$). Prove that, for every $q \in M \setminus B(p, r)$, there exists a point $q' \in \partial B(p, r)$ such that $d(p, q) = r + d(q', q)$.

3. Generalize the first variation formula (Theorem 5.3, Chapter IV in the lecture notes) to the case of a variation that is not necessarily proper.

4. Suppose that $N$ and $N'$ are submanifolds of $M$ and that $\gamma : [0, d] \to M$ is a unit speed geodesic such that $\gamma(0) \in N$, $\gamma(d) \in N'$, and that $\ell(\gamma) = d = d(N, N') > 0$. Here $d(N, N') = \inf \{d(x, y) : x \in N, y \in N'\}$. (In other words, $\gamma$ minimizes the distance between $N$ and $N'$.) Show that $\dot{\gamma}_0 \perp T_{\gamma(0)} N$ and $\dot{\gamma}_d \perp T_{\gamma(d)} N'$. [Use Problem 3.]

5. Let $A_{ij} : \mathbb{R}^m \to \mathbb{R}$, $i, j = 1, \ldots, n$, be smooth mappings and denote $A = (A_{ij})$. Prove that in the open set $\{x \in \mathbb{R}^m : \det A > 0\}$ we have

$$\frac{\partial}{\partial x^k} \log \det A = \text{tr} \left( \frac{\partial A}{\partial x^k} A^{-1} \right)$$

for all $k = 1, \ldots, m$. 
