1. Let $\gamma : I \to M$ be a $C^\infty$-path. For $t_0, t \in I$, define a mapping (linear isomorphism) $P_{t_0,t} : T_{\gamma(t_0)}M \to T_{\gamma(t)}M$ by $P_{t_0,t}v = V(t)$, where $V \in T(\gamma)$ is the parallel transport of $v \in T_{\gamma(t_0)}M$ along $\gamma$. Prove that

$$D_tW(t_0) = \lim_{t \to t_0} \frac{P_{t_0,t}^{-1}W(t) - W(t_0)}{t - t_0}$$

for $W \in T(\gamma)$. [Hint: Use a parallel frame along $\gamma$.]

2. Let $M$ be a Riemannian manifold and let $U \subset M$ be an open set. The divergence of a vector field $X \in T(U)$, denoted by $\text{div} X$, is the trace of the linear map $Y \mapsto \nabla_Y X$. Thus $\text{div} : U \to \mathbb{R},$ 

$$(\text{div} X)(p) = \text{tr}(v \mapsto \nabla_v X), \quad v \in T_p M.$$ 

Suppose that $(U, x)$, $x = (x^1, \ldots, x^n)$, is a chart. Express $\text{div} X$ in local coordinates. [Recall that the trace of an $n \times n$ matrix $(a_{ij})$ is the sum of the diagonal entries $\sum_{i=1}^n a_{ii}$.]

3. Let $M$ be a Riemannian manifold, $\langle \cdot, \cdot \rangle$ the Riemannian metric, and $\nabla$ the Riemannian connection of $M$. The Hessian of a real-valued function $u \in C^\infty(M)$ is a 2-covariant tensor field $\text{Hess} f \in T^2(M)$ defined by

$$\text{Hess} f(X, Y) = \langle \nabla_X (\nabla f), Y \rangle, \quad X, Y \in T(M).$$

Prove that $\text{Hess} f$ is symmetric and

$$\text{Hess} f(X, Y) = X(Y f) - (\nabla_X Y)f.$$

4. (a) Prove that the mapping $L : T(M) \times T(M) \to T(M)$,

$$L(X, Y) = L_X Y$$

(= the Lie derivative of $Y$ with respect to $X$) is not a connection.

(b) Prove that there exist smooth vector fields $V \in T(\mathbb{R}^2)$ and $W \in T(\mathbb{R}^2)$ such that $V = W = \frac{\partial}{\partial x}$ along the $x$-axis, but the Lie derivatives $L_V \left( \frac{\partial}{\partial y} \right)$ and $L_W \left( \frac{\partial}{\partial y} \right)$ are not equal on the $x$-axis. (Conclusion?)

5. Let $Y \in T(M)$ be a vector field on a Riemannian manifold $M$ such that $|Y|$ is constant. Prove that $\nabla_X Y$ is always perpendicular to $Y$ (i.e. $\langle \nabla_X Y, Y \rangle = 0$) for all $X \in T(M)$.

**Note:** Recall, for instance from the course "Introduction to differential geometry", that $L_X Y = [X, Y]$. 