1. Let $M$ and $N$ be differentiable manifolds and $f: M \to N$. Suppose that for every $p \in M$ there exist a chart $(U, x)$ in $M$ and a chart $(V, y)$ in $N$ such that $p \in U$, $f U \subset V$, and $y \circ f \circ x^{-1} \in C^\infty(xU)$. Prove that $f$ is $C^\infty$.

2. (a) Let $v, w \in T_p M$ and $c, d \in \mathbb{R}$. Verify that the mappings 
\[ (cv + dw): C^\infty(p) \to \mathbb{R}, \]
\[ (cv + dw)(f) = cv(f) + dw(f), \]
is a tangent vector in $p$.
(b) Verify that coordinate vectors $(\partial_i)_p$ are tangent vectors at $p$.
(c) Let $x = (x^1, \ldots, x^n)$ be a chart at $p$ and let $(\partial_i)_p$, $i = 1, \ldots, n$, be the corresponding coordinate vectors. Show that 
\[ (\partial_i)_p x^j = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases} \]

3. Let $M$ be a differentiable manifold, $p \in M$, $I \subset \mathbb{R}$ an open interval, and $0 \in I$.
(a) Let $\gamma: I \to M$ be a $C^\infty$-path such that $\gamma(0) = p$. Show that $\dot{\gamma}_0 \in T_p M$.
(b) Let $v \in T_p M$. Show that there exists a $C^\infty$-path $\gamma: I \to M$ such that $\dot{\gamma}_0 = v$.

4. Let $M$ be a differentiable manifold, $p \in M$, and $v \in T_p M$. Prove: If $f \in C^\infty(p)$ is a constant function, then $vf = 0$.

5. Let $M$, $N$, and $L$ be differentiable manifolds and let $f: M \to N$ and $g: N \to L$ be $C^\infty$-mappings. Show that 
\[ (g \circ f)_p = g_{\ast f(p)} \circ f_p \]
for all $p \in M$. 