Towards a proof of Fourier’s law

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Hamiltonian dynamics

Hamiltonian dynamics: \((q, p) \in \mathbb{R}^{2N}, H : \mathbb{R}^{2N} \to \mathbb{R}\)

\[
\dot{q} = p, \quad \dot{p} = -\frac{\partial H}{\partial q}
\]

What are "natural" measures \(\mu\)?

- Closed system: \(m(dpdq) \) or \(m \big|_{H=\text{const}}\)
- Open system, equilibrium (Gibbs) measure:

\[
\mu^T = e^{-H(p,q)/T} m(dpdq)
\]

For nonequilibrium stationary states no such simple formula.
Equilibrium vs. non-equilibrium

Open system: interaction with an environment:

- Eg. add noise $\rightarrow$ Markov process
- Look for a (hopefully unique) stationary measure
  $= "\text{nonequilibrium stationary state}"$
- "Uniform" noise: stationary measure is Gibbs measure $\mu^T ("\text{equilibrium}" \text{ state})$

Return to equilibrium of a closed system

- Let $\mu_0$ be "close" to a Gibbs measure $\mu^T$
- Let $\mu_t = \phi_t \mu_0$, $\phi_t$ Hamiltonian flow
- Show $\mu_t \rightarrow \mu^T$ as $t \rightarrow \infty$.
- This can be strictly true only for infinite dimensional systems.
Extended dynamics

Extended dynamical system:

- spatial structure, volume $V$
- dimension of attractor $\to \infty$ as $V \to \infty$

Examples

- Parabolic PDE’s, data in $L^\infty(\Lambda), |\Lambda| \to \infty$
- $N$-particle Hamiltonian systems, $N \to \infty$
- Discrete time models: Coupled map lattices

New phenomena vs. finite dimensional dynamics:

- Spatial patterns
- Transport and diffusion
Transport and diffusion

**NESS**: extended Hamiltonian system in a box, noise on the boundary

\[ T_1 = T_2: \text{equilibrium} \]
\[ T_1 \neq T_2: \text{energy flux} \]

**Approach to equilibrium**:

- Box = \( \mathbb{R}^d \), \( \mu_0 \sim \mu^T \) at spatial infinity
- Show \( \mu_t \to \mu^T \) diffusively as \( t \to \infty \)
Coupled dynamics

Models: **Coupled maps** and **Coupled flows**

- Subsystems (maps or flows) indexed by $x \in \mathbb{Z}^d$
- Couple together locally

**Weakly anharmonic coupled oscillators:**

- Lots of numerics.
- In a weak anharmonicity scaling limit (**kinetic limit**) get formally a Boltzmann equation (Spohn)
- Diffusion and Fourier proved there (J.B., A.K. math-ph 0703014)
- Hard to prove kinetic limit and even heuristically to study corrections.
Coupled chaotic systems

Weakly coupled strongly chaotic systems

- Bunimovich, Liverani, Pellegrinotti, Suhov, Eckmann, Young ...
Local energy and flux

Total energy is sum of local energies $H_x$:

$$H = \sum_x H_x$$

No coupling: each $H_x$ is conserved, $\dot{H}_x = 0$.

Turn on coupling: only $H$ is conserved and

$$\dot{H}_x = -\nabla \cdot J_x$$

$J_x$ flux of energy at site $x$.

Show: $H_x$ diffuses and $J_x$ is tied to to $H_x$ by Fourier’s law.
Return to equilibrium

Infinite system \( V = \mathbb{Z}^d \), \( \mu_t = \phi_t \mu_0 \)

Diffusion in **hydrodynamic limit**:

- Take \( \mu_0 \) s.t. \( E_{\mu_0} H_x = \tau(\epsilon x) \)
- Let

\[
\tau(t, x) := \lim_{\epsilon \to 0} E_{\mu_{t/\epsilon^2}} H_x / \epsilon
\]

\[
j(t, x) := \lim_{\epsilon \to 0} \frac{1}{\epsilon} E_{\mu_{t/\epsilon^2}} J_x / \epsilon
\]

- Show:

\[
j = -\kappa(\tau) \nabla \tau \quad \text{Fourier law}
\]

\[
\dot{\tau} = \nabla \cdot (\kappa(\tau) \nabla \tau) \quad \text{Diffusion}
\]

- Actually, these should hold **almost surely** in \( \mu_0 \)
NESS

Fix $\Lambda \subset \mathbb{R}^d$ and $t : \partial \Lambda \rightarrow \mathbb{R}^+.$

Lattice box $V = \Lambda/\epsilon \cap \mathbb{Z}^d,$

- Hamiltonian dynamics in $V$
- noise of energy $T_x = t_x/\epsilon$ at $x \in \partial V.$

For $x \in \Lambda$ let

$$j_x = \frac{1}{\epsilon} E J_x/\epsilon \quad \tau_x = E H_x/\epsilon$$

$E$ expectation in NESS

Then, as $\epsilon \rightarrow 0$ show

- **Fourier law**: $j(x) = -\kappa(\tau(x)) \nabla \tau(x)$
- **Temperature profile**:

$$\nabla \cdot (\kappa(\tau) \nabla \tau) = 0$$

$$\tau|_{\partial \Lambda} = t$$
**Coupled map lattice**

**Coupled map lattice** is a discrete space time PDE. 

Phase space

\[ \mathcal{M} = \times_{x \in \mathbb{Z}^d} M_x \]

\(M_x\) copies of a manifold \(M\).

Let \(\phi : M \to M\) be the **local dynamics**.

**Uncoupled** dynamics is given by the product \(\Phi : \mathcal{M} \to \mathcal{M}\)

\[ \Phi(m)_x = \phi(m_x) \]

The role of **Laplacean** is played by **coupling**

\[ \psi : \mathcal{M} \to \mathcal{M} \]

a diffeomorphism which is **local**: \(\psi(m)_x\) depends weakly on \(m_y\) for \(|y - x|\) large.

The **CML** dynamical system is \((\mathcal{M}, \psi \circ \Phi)\).
Coupled map lattice with a conservation law

We take $M = \mathbb{R}_+ \times N$ and

$$\phi(E, \theta) = (E, f(\theta)), \quad E \in \mathbb{R}, \theta \in N$$

with $f$ hyperbolic, e.g. $N = S^1$, $f(\theta) = 2\theta$

- Energy of each cell is conserved: $E_x \rightarrow E_x$ i.e. one vanishing Lyapunov exponent per unit volume.
- Chaotic dynamics for the rest: $\theta_x \rightarrow f(\theta_x)$
- Coupling typically removes the degeneracy
- Look for coupling so that total energy $E = \sum_x E_x$ is conserved.
Coupling

**Coupling**: nearby cells interact, exchange energy

\[
E'_y = \sum_{|x-y|=1} p_{xy}(E, \theta)E_x
\]

\[
\theta'_x = f(\theta_x) + g_x(E, \theta)
\]

- \(p_{xy}, g_x\) depend on \(\theta_u, E_u\) for \(u\) near \(x\) only
- \(p_{xy} \geq 0\)
- \(\sum_y p_{xy}(E, \theta) = 1\) for all \(E, \theta\)

**Total energy** \(\sum_x E_x\) conserved.
Diffusion

Let, at \( t = 0 \), \( E_x \rightarrow T \) as \( |x| \rightarrow \infty \).
Show \( E_x(t) \) diffuses to \( T \) almost surely in \( \theta(0) \)

\[
E_x(t) - T \sim t^{-d/2} f(x/\sqrt{t})
\]

Hydrodynamic scaling limit:

- Let \( E_x(0) = \tau(\epsilon x) \)
- Show: \( \lim_{\epsilon \rightarrow 0} E(t/\epsilon^2, x/\epsilon) = \tau(t, x) \) satisfies

\[
\dot{\tau} = \nabla \cdot (\kappa(\tau) \nabla \tau)
\]

almost surely in \( \theta(0) \).
Random walk

Iteration of coupled maps

\[ E_y(t + 1) = \sum_{|x-y|=1} p_{xy}(E(t), \theta(t)) E_x \]
\[ \theta_x(t + 1) = f(\theta_x(t)) + g_x(E(t), \theta(t)) \]

where \( p_{xy} \geq 0 \) and

\[ \sum_y p_{xy}(E, \theta) = 1. \]

- \( p_{xy}(E(t), \theta(t)) := p_{xy}(t) \) can be viewed as transition probabilities of a random walk
- \( E_x(t) \) is (proportional to) the probability of finding the walker at \( x \) at time \( t \)
Random walk in random environment

- Transition probabilities $p_{xy}(t)$ depend on space and time: random walk in a space-time dependent environment
- $p_{xy}(t)$ completely determined by initial conditions of $E, \theta$
- Prove: walk is diffusive almost surely in $\theta|_{t=0}$
- Typical $\theta|_{t=0} \rightarrow$ random $p_{xy}(t)$
- Prove quenched CLT for such walks i.e. a.s. in the $p$-ensemble

What is the statistics of $p$ like?
Suppose first $p_{xy}$ and $g_x$ depend only on $\theta$. Then:

- $f$ hyperbolic, $g$ small, smooth $\implies f + g$ hyperbolic
- Coupled Map Lattice $\implies \theta$-dynamics space time mixing $\implies p_{xy}(t)$ weakly correlated in space and time
- Use Renormalization to prove randomness is irrelevant $\implies$
- Random walk satisfies CLT a.s. in $p$ $\implies$
- $E$ diffuses almost surely in $\theta_{t=0}$.
Slow and fast variables

Suppose $p_{xy}$ depends on $E$ too

- $\theta$ still space time mixing
- $p(\theta(t), E(t))$ gets \textbf{long range correlations} through $E$ dependence
- RG $\Rightarrow$ $E$ dependence \textbf{irrelevant} $\Rightarrow$
- CLT still holds

Suppose also $g_x$ depends on $E$

- \textbf{Fast} variables $\theta$ get slaved to the slow ones $E$

Hard problem of deterministic diffusion reduced to an easier one on RW in weakly correlated environment
Diffusion

Random walk with space time dependent transition probability \( p_{xy}(t) \).

Probability of a walk \( \omega = (\omega_0, \ldots, \omega_T) \) in time \( T \)

\[
P^T(\omega) = \prod_{t=0}^{T-1} p_{\omega_t \omega_{t+1}}(t).
\]

\( E_T \) expectation in walks with \( \omega_0 = 0 \). Diffusion constant

\[
D_T = T^{-1} E_T \omega(T)^2
\]

Diffusion:

\[
\lim_{T \to \infty} D_T = D
\]
Scaling limit

Rescale to space $\Omega$ of paths $\omega : [0, 1] \to \mathbb{R}^d$

$$\omega(t) = T^{-\frac{1}{2}} \omega_T$$

$E_T$ induces expectation $\mathcal{E}_T$ on such paths

$$\mathcal{E}_T F(\omega(\cdot)) = E_T F(T^{-\frac{1}{2}} \omega_T).$$

Scaling limit

$$\lim_{T \to \infty} \mathcal{E}_T F := \mathcal{E} F$$

for $F : \Omega \to \mathbb{R}$ continuous on path space.

Prove: **allmost surely** in the $p$ ensemble $\mathcal{E}$ exists and equals Wiener measure, diffusion constant $D$:

$$D = \lim_{T \to \infty} D_T = \mathcal{E} \omega(1)^2$$
Renormalization group

Probability to walk from $x$ to $y$ during time interval $[t, t']$:

$$P_{t,t'}(x, y, p) = (p(t) \ldots p(t' - 1))_{xy}$$

Define renormalized transition probability matrix

$$(R_l p)_{xy}(t) = l^d P_{l^2 t, l^2 (t+1)}(lx, ly, p)$$

for walks on $l^{-1} \mathbb{Z}^d$. Then, if $l^2$ divides $t, t'$,

$$P_{t,t'}(x, y, p) = l^{-d} P_{t/l^2, t'/l^2}(l^{-1} x, l^{-1} y, R_l p).$$

$R_l p$ is the Renormalization group flow in a space of random matrices.
Asymptotics

Scaling limit controlled by $R_l$ as $l \to \infty$

- Diffusion constant at time $t$:

$$D(t, p) = t^{-1} \sum_x P_{0,t}(0, x, p)x^2$$

reduces to **unit time** one with rates $R_lp$:

$$D(l^2, p) = D(1, R_lp).$$

- Let $F : \Omega \to \mathbb{R}^d$ depend on $\omega$ restricted to $\tau^{-1}\mathbb{Z}$ and $l^2 = T/\tau$. Then

$$\mathcal{E}_TF(\omega(\cdot)) = E_{\tau}^{R_lp}F(\tau^{-\frac{1}{2}}\omega_{\tau.})$$

= **fixed time** $\tau$ problem with rates $R_lp$. 
Asymptotics

Scaling limit controlled by $R_l$ as $l \to \infty$

- Diffusion constant at time $t$:

$$D(t, p) = t^{-1} \sum_x P_{0,t}(0, x, p)x^2$$

reduces to **unit time** one with rates $R_l p$:

$$D(l^2, p) = D(1, R_l p).$$

- Let $F : \Omega \to \mathbb{R}^d$ depend on $\omega$ restricted to $\tau^{-1}\mathbb{Z}$ and $l^2 = T/\tau$. Then

$$\mathcal{E}_T F(\omega(\cdot)) = E^{R_l p}_\tau F(\tau^{-\frac{1}{2}} \omega_\tau).$$

= **fixed time** $\tau$ problem with rates $R_l p$. 
Fixed point

If \( R_l p \to p^* \) as \( l \to \infty \) then

\[
D = D(1, p^*)
\]

and scaling limit is given by

\[
\mathcal{E} F(\omega(\cdot)) = E_{\tau}^{p^*} F(\tau^{-\frac{1}{2}} \omega_{\tau} \cdot).
\]

Convergence to Wiener measure:

\[
p_{x,y}^* = (2\pi D)^{-d/2} e^{-\frac{(x-y)^2}{2D}}.
\]

Thus, we want to prove \( R_l p \) becomes nonrandom as \( l \to \infty \) and converges to \( p^* \) a.s. in \( p \).
Semigroup

$R_l$ satisfies $R_{ll'} = R_l R_{l'}$.

Study $R_l$ iteratively:

- Pick $L > 1$ and let $R := R_L$
- Let $p_n = R^n p$ i.e. $p_n = R_L^n p$
- Let $E$ be expectation in $p$ ensemble. Write

  $$p_n = E p_n + b_n.$$

- Show
  - $b_n \to 0$ almost surely
  - $E p_n \to p^*$. 
Assumptions

Assume

- Distribution of $\rho$ translationally and rotationally invariant
- $E\rho_{xy} = T(x - y)$ exponentially decaying
- Cumulants of $\rho$ cluster exponentially

$$E(p_{x_1 y_1}(t_1); p_{x_2 y_2}(t_2); \ldots; p_{x_N y_N}(t_N)) \leq \epsilon^N e^{-\lambda \tau},$$

$\tau$ length of shortest tree on the space time support

Assumptions are satisfied by

- $\rho(\theta)$ analytic, local with $\theta$ analytic CML
- $\rho(s)$ local in spins of a high temperature Ising model and the like
Result

Theorem: In all dimensions $d \geq 1$

$$E(R^n p_{x_1 y_1}(t_1); R^n p_{x_2 y_2}(t_2); \ldots; R^n p_{x_N y_N}(t_N)) \leq \epsilon_n^N e^{-\lambda \tau},$$

with $\epsilon_n \to 0$ as $n \to \infty$ (exponentially).

Randomness irrelevant in all dimensions

Implications: $D$ exists, scaling limit Wiener
General case

Include $E$ dependence of the walk

$$E_y(t + 1) = \sum_{|x-y|=1} p(\theta(t), E(t))_{xy} E_x(t)$$

Environment depends on the trajectory $E(t)$. $\rho(s), \rho(t)$ **diffusively** correlated.

RG for **conditional** transition probabilities.

- Let $E_n(t, x) = L^{nd} E(L^{2nt}, L^n x)$
- $p_n(t) = p_n(t, E_n(t))$ is conditioned on $E_n(t)$ i.e. collects rescaled walks on time interval $[L^{2nt}, L^{2n}(t + 1)]$ conditioned on $E(L^{2nt})$.
- $\{p_n(t, E)\}$, $E$ fixed **exponentially** weakly correlated
- $E$ dependence an **irrelevant** perturbation in the RG.
Analogy in continuum

Our CML is a discrete version of the SPDE

\[ \dot{E} = \partial_\mu (a_{\mu\nu}(E) \partial_\nu E + b_\mu(E)E) \]

where \( a_{\mu\nu} \) and \( b_\mu \) are random and nonlinear

The RG produces a non random PDE in the scaling limit

\[ \dot{E} = \partial_\mu (\kappa_{\mu\nu}(E) \partial_\nu E) \]

It is a combination of RG for PDE’s (J.B. & A.K., 1992) and RG for RWRE (J.B. & A.K., 1991).

However, the randomness is deterministic
What kind of CML should model the coupled billiards?

**Rare configurations** of $E$ can **slow down** mixing of energies and $\theta$ dynamics $\implies$

- $p(\theta, E)$ may get close to 1 or 0
- Correlation times for $p(\theta, E)$ can blow up

These issues can be studied with the RG
Conclusions

Rather general class of CML with a conservation law can be studied with the RG

A new approach to study hydrodynamic limits of particle systems, interacting random walks etc.

Challenge: real Hamiltonian systems
Linearized RG

Given \( p = \{p(t)_{xy}\} \), compute \( Rp = \{p'(t')_{x'y'}\} \) from

\[
p'(t')_{x'y'} = L^d(p(L^2 t') \ldots p(L^2(t' + 1) - 1))_{Lx'Ly'}
\]

Write

\[ p_{xy} = T(x - y) + b_{xy} \]

with \( Ep = T \) and \( Eb = 0 \).

Let \( Rp = T' + b' \). To linear order in \( b \)

\[ T'(x' - y') = L^d T^{L^2} (Lx' - Ly') \]

and (let \( t' = 0 \))

\[
b'_{x'y'} = L^d \sum_{t=1}^{L^2} \sum_{xy} T^t (Lx' - x)b_{xy}(t) T^{L^2-t-1} (y - Ly').
\]
Contraction

Let \( \hat{T}(k) = 1 - c k^2 + o(k^2) \). As \( L \to \infty \):

\[
\hat{T}'(k) = \hat{T}^{L^2}(k/L) \to e^{-ck^2} = \hat{p}^*(k)
\]

For \( b \) use \( \sum_y p_{xy} = 1 \) implying

\[
\sum_y b_{xy} = 0
\]

to get

\[
b'_{x'y'} \sim L^d \sum_{t=1}^{L^2} \sum_{xy} T^t(Lx' - x)b_{xy}(t) \nabla_y T^{L^2-t-1}(y - Ly')
\]
Contraction

For $t = \mathcal{O}(L^2)$,

$$T^t(Lx' - x) \sim L^{-d} e^{-(x' - x/L)^2}$$

$$\nabla_y T^{L^2-t-1}(y - Ly') \sim L^{-d-1} e^{-(y' - y/L)^2}$$

so e.g.

$$b'_{00}(0) \sim L^d L^{-d} L^{-d-1} \sum_{t<L^2} \sum_{|x|<L} b_{xx}(t)$$

so since $b_{xx}(t) \sim \text{i.i.d.}$

$$E(Rb)^2 \sim L^{-2d-2} L^{d+2} Eb^2 = L^{-d} Eb^2$$

Noise is irrelevant in all dimensions.