GEOMETRY of GERBES
and
CONFORMAL FIELD THEORY

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- A bit of history
- Why (abelian) gerbes?
- Local data
- Gerbes on Lie groups and WZW models
- Quantization
- Gauge fields twisted by gerbes and branes
- Gerbes and orientifolds
- Conclusions
A bit of history:

non abelian cohomology and abstract gerbes
(\textit{Giraud 1971})

Deligne cohomology
(\textit{Deligne, Beilinson, mid 1980's})

Deligne cohomology in topological field theory and WZW models
(\textit{K.G. 1987})

loop spaces, characteristic classes and geometric quantization
(\textit{Brylinski 1991})

(line-)bundle gerbes
(\textit{Murray 1994, Murray-Stevenson 1999})

gerbes, anomalies and index
(\textit{Carey-Mickelsson-Murray 1995})
gerbes on Lie groups
(Chatterjee-Hitchin 1998)
(Brylinski 2000)
(Meinrenken 2002)
(K.G.-Reis 2003)

gerbe modules, twisted gauge fields and branes
(Kapustin 1999)

gerbes, gerbe modules and twisted K-theory
(Bouwknegt-Carey-Mathai-Murray-Stevenson 2001)
(Mickelsson et al. 2002-)

gerbe modules and WZW branes
(K.G.-Reis 2002)
(K.G. 2004)

gerbes and orientifolds
(Schreiber-Schweigert-Waldorf 2006)
Line bundles (with connection)

- For an exact 2-form $F = dA$ on $M$,
  $$\int_S F = \int_{\partial S} A \quad \text{Stokes Theorem}$$

- If $F$ is closed with periods $\in 2\pi \mathbb{Z}$ then
  $$\int_S F = \frac{1}{\sqrt{-1}} \log \text{hol}_\mathcal{L}(\partial S) \mod 2\pi$$
  if $\mathcal{L}$ is a line bundle of curvature $F$ with $\text{hol}_\mathcal{L}(\ell)$ denoting
  the holonomy along the closed curve $\ell : S^1 \to M$

- For flat line bundles $\mathcal{L}$ with $F = 0$
  $$\text{hol}_\mathcal{L}(\cdot) \in \pi_1(M)^* = H^1(M, U(1))$$
  and it determines the isomorphism class $[\mathcal{L}]$ of $\mathcal{L}$

- For fixed $F$, $[\mathcal{L}' \otimes \mathcal{L}^*] \in H^1(M, U(1))$
(Line-bundle) gerbes (with connection)

- For an exact 3-form \( H = dB \) on \( M \),

\[
\int_{\Sigma} H = \int_{\partial V} B \quad \text{Stokes Theorem}
\]

- If \( H \) is closed with periods \( \in 2\pi \mathbb{Z} \) then

\[
\int_{\Sigma} H = \frac{1}{\sqrt{-1}} \log \text{hol}_G(\partial V) \mod 2\pi
\]

if \( G \) is a gerbe of curvature \( H \) with \( \text{hol}_G(\phi) \) denoting

the holonomy along the closed surface \( \phi : \Sigma \to M \)

- For flat gerbes \( G \) with \( H = 0 \)

\[
\text{hol}_G(\cdot) \in H^2(M, U(1))
\]

and it determines the isomorphism class \([G]\) of \( G \)

- For fixed \( H \), \([G' \otimes G^*] \in H^2(M, U(1))\)
Local data

Let \((\mathcal{O}_i)\) be a (good) open covering of \(M\)

- \(\mathcal{L}\) is represented by \((A_i, g_{ij})\) s.t.
  \[
  F = dA_i \quad \text{on } \mathcal{O}_i ,
  \]
  \[
  A_j - A_i = \sqrt{-1} \, d \log g_{ij} \quad \text{on } \mathcal{O}_{ij} \equiv \mathcal{O}_i \cap \mathcal{O}_j ,
  \]
  \[
  g_{ij} g_{ik}^{-1} g_{jk} = 1 \quad \text{on } \mathcal{O}_{ijk}
  \]

- \(\mathcal{G}\) is represented by \((B_i, A_{ij}, g_{ijk})\) s.t.
  \[
  H = dB_i \quad \text{on } \mathcal{O}_i ,
  \]
  \[
  B_j - B_i = dA_{ij} \quad \text{on } \mathcal{O}_{ij} ,
  \]
  \[
  A_{ij} - A_{ik} + A_{jk} = \sqrt{-1} \, d \log g_{ijk} \quad \text{on } \mathcal{O}_{ijk} ,
  \]
  \[
  g_{ijk} g_{ijl} g_{ikl} g_{jkl}^{-1} = 1 \quad \text{on } \mathcal{O}_{ijkl}
  \]
Physical interpretation:

- $A(\ell) = e^{-\frac{1}{2} \|d\ell\|^2} \text{hol}_L(\ell)$ is the Feynman amplitude of a particle in the electromagnetic field $F$

- $A(\phi) = e^{-\frac{1}{2} \|d\phi\|^2} \text{hol}_G(\phi)$ is the Feynman amplitude of a closed string in the Kalb-Ramond field $H$
Example:

\[ M = G \] is a simple compact \textbf{Lie group}

\[ H_k = \frac{1}{12} \left\langle g^{-1}dg, [g^{-1}dg, g^{-1}dg] \right\rangle_k \] is a bi-invariant 3-form

where \( g^{-1}dg \) is the Maurer-Cartan form and \( \left\langle \cdot, \cdot \right\rangle_k \) is the Killing form on \( g = \text{Lie}(G) \) normalized by \( \left\langle \alpha^\vee, \alpha^\vee \right\rangle_k = \frac{k}{\pi} \) for short coroots

The number \( k \) is called the \textbf{level}

For \( G \) \textbf{simply connected}:

The periods of \( H_k \) are in \( 2\pi \mathbb{Z} \) iff \( k \in \mathbb{Z} \). The corresponding gerbe \( G^k \) is unique (modulo isom.). It was \textbf{constructed}:

- for \( SU(2) \) by \textit{K.G. (1986)}
- for \( SU(N) \) by \textit{Chatterjee-Hitchin (1998)}
- for \( G \) compact simply connected by \textit{Meinrenken (2002)}
For $G$ non simply connected (K.G.-Reis 2003):

Let $G = \tilde{G}/Z$ for $Z \subset \text{center}(\tilde{G})$ and $\tilde{G}$ simply connected

- There is an obstruction $[u] \in H^3(Z, U(1))$ to the existence of gerbe $G^k$ on $G$ with curvature $H_k$

- Triviality of $[u]$ selects the levels $k$ s.t. $G^k$ exists

- $G^k$ is then unique except for $G = SO(4n)/\mathbb{Z}_2$ with $G^k_{\pm}$ (discrete torsion)

Wess-Zumino-Witten (WZW) model

$\equiv$ string in a group manifold $G$ with the Feynman amplitude involving $\text{hol}_{G^k}$

- A prototype example of the Conformal Field Theory
Geometric quantization of the bulk (≡ closed string) WZW theory

• By transgression
  
  gerbe $\mathcal{G}$ on $M$  
  line bundle $\mathcal{L}_g$ on the loop space $LM$

• The space of quantum states of the bulk WZW theory:
  
  $\mathcal{H} = \Gamma(\mathcal{L}_g^k)$  
  space of sections

  with a geometric action of the double affine algebra $\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}$ of level $k$

• The spectrum of irreducible highest weight reps. $\hat{V}_\lambda$ of $\hat{\mathfrak{g}}$
  
  $\mathcal{H} = \bigoplus_{\lambda, \lambda'} M_{\lambda \lambda'} \otimes \hat{V}_\lambda \otimes \hat{V}_{\lambda'}$

  found by identifying the highest weight subspaces $M_{\lambda \lambda'} \subset \Gamma(\mathcal{L}_g^k)$

  $\longrightarrow$ partition functions (Felder-K.G.-Kupiainen 1987)
Gauge fields twisted by gerbes (Kapustin 1999)

Let $\mathcal{G}$ be a gerbe on $M$ with local data $(B_i, A_{ij}, g_{ijk})$

- A vector bundle $\mathcal{E}$ twisted by $\mathcal{G}$ with a gauge field is given by $n \times n$-matrix valued local data $(A_i, G_{ij})$ s.t.
  
  $A_j = G_{ij}^{-1} A_i G_{ij} - \sqrt{-1} G_{ij}^{-1} dG_{ij} + A_{ij} 1 = 0$ on $\mathcal{O}_{ij}$
  
  $G_{ij} G_{jk} = g_{ijk} G_{ik}$ on $\mathcal{O}_{ijk}$

- $\mathcal{E}$ is also called a $\mathcal{G}$-module (of rank $n$)

- Let $W_{\mathcal{E}}(\ell) = \text{tr}(\text{hol}_{\mathcal{E}}(\ell))$ be the Wilson loop “observable”
  Due to the twist by $\mathcal{G}$ the phase of $W_{\mathcal{E}}(\ell)$ is ambiguous

- **But:** if $\phi : \Sigma \rightarrow M$ with $\partial \Sigma = \bigsqcup S_\alpha$ and $\mathcal{E}_\alpha$ are $\mathcal{G}$-modules then for $\ell_\alpha = \phi|_{S_\alpha}$
  
  $\text{hol}_G(\phi) \prod_\alpha W_{\mathcal{E}_\alpha}(\ell_\alpha)$, is unambiguous !!!
Problem with the $\mathcal{G}$-modules: they exist only if the curvature $H$ of $\mathcal{G}$ is exact !!!

Solution:

A pair $\mathcal{D} = (D, \mathcal{E})$ is called a $\mathcal{G}$-brane if $D \subset M$ and $\mathcal{E}$ is a $\mathcal{G}|_D$-module.

If $\phi : \Sigma \to M$ and $\phi(S_\alpha) \subset D_\alpha$ for $S_\alpha \subset \partial \Sigma$ then the product $\text{hol}_{\mathcal{G}}(\phi) \prod_\alpha W_{\mathcal{E}_\alpha}(\ell_\alpha)$ is well defined and

$$ A(\phi) = e^{-\frac{1}{2} \|d\phi\|^2} \text{hol}_{\mathcal{G}}(\phi) \prod_\alpha W_{\mathcal{E}_\alpha}(\ell_\alpha), $$

gives the Feynman amplitude of an open string with ends moving in the branes $D_\alpha$ and coupled to the (twisted) “Chan-Paton” gauge fields.
Example of the WZW model

- The $\mathcal{G}^k$-branes $\mathcal{D} = (D, \mathcal{E})$ are called symmetric if they conserve the diagonal symmetry $\hat{g} \subset \hat{g} \oplus \hat{g}$. They have the conjugacy classes

$$D = \mathcal{C}_\lambda \equiv \{ h e^\lambda h^{-1} | h \in G \}$$

as supports and the curvature of $\mathcal{E}$ given by the 2-form

$$F = 2 \left\langle h^{-1} dh, \text{Ad}_{e^\lambda} (h^{-1} dh) \right\rangle_k$$

($\lambda$ is a weight viewed as an element of $g$)

Classification of sym. $\mathcal{G}^k$-branes (K.G. 2004):

- For simply connected $G$:

$$\mathcal{D} = (\mathcal{C}_\lambda, n \mathcal{E}^1)$$

with $(\mathcal{C}_\lambda, \mathcal{E}^1)$ a unique symmetric brane of rank 1 supported by $\mathcal{C}_\lambda$ and $n \mathcal{E}^1 \equiv \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}^1$ $n$ times
• For non simply connected $G = \tilde{G}/Z$

$$\mathcal{D} = (\mathcal{C}_\lambda, n_1 \mathcal{E}_1^1 \oplus \cdots \oplus n_\ell \mathcal{E}_{n_\ell}^1)$$

with $|Z_\lambda|$ different symmetric branes $(\mathcal{C}_\lambda, \mathcal{E}_i^1)$ of rank 1 supported by $\mathcal{C}_\lambda \cong \tilde{\mathcal{C}}_\lambda/Z_\lambda$ for $Z_\lambda \subset Z$ with the holonomy differing by the characters of $Z_\lambda \cong \pi_1(\mathcal{C}_\lambda)$

**Example:**

• For $G = SU(2) \cong S^3$ the admitted conjugacy classes $\mathcal{C}_j$ correspond to spins (weights) $j = 0, \frac{1}{2}, \ldots, \frac{k}{2}$

They are the spheres $S^2$ under the angles $\frac{2\pi j}{k}$

Each carries one symmetric $G^k$-brane of rank 1

• For $G = SO(3) \cong \mathbb{R}P^3$ level $k$ has to be even

Are admitted the conjugacy classes $\mathcal{C}_j \cong \tilde{\mathcal{C}}_j$ for $j < \frac{k}{4}$ carrying one rank 1 symmetric $G^k$-brane and $C_{\frac{k}{4}} \cong \tilde{\mathcal{C}}_{\frac{k}{4}}/\mathbb{Z}_2$ carrying two symmetric rank 1 $G^k$-branes
• In principle there is an obstruction $[v] \in H^2(Z_\lambda, U(1))$ to the existence of rank 1 $\mathcal{G}^k|_{\mathcal{C}_\lambda}$-module $\mathcal{E}^1$ on $\mathcal{C}_\lambda \cong \tilde{\mathcal{C}}_\lambda/Z_\lambda$ but $H^2(Z_\lambda, U(1)) = 0$ for cyclic groups $Z_\lambda = \mathbb{Z}_m$

**Exceptional case:**

• For $G = SO(4n)/\mathbb{Z}_2 = Spin(4n)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ and $\mathcal{C}_\lambda \cong \tilde{\mathcal{C}}_\lambda/Z_\lambda$ with $Z_\lambda = \mathbb{Z}_2 \times \mathbb{Z}_2$ the obstruction $[v] \in H^2(Z_\lambda, U(1)) = \mathbb{Z}_2$ forbids the existence of a rank 1 $\mathcal{G}^k|_{\mathcal{C}_\lambda}$-module and

$$\mathcal{D} = (\mathcal{C}_\lambda, n\mathcal{E}^2)$$

where $(\mathcal{C}_\lambda, \mathcal{E}^2)$ is a unique symmetric rank 2 $\mathcal{G}^k$-brane

$\longrightarrow$ generation of a **non-abelian gauge symmetry**
Geometric Quantization of boundary (≡ open string) WZW theory:

- By transgression

  \[ \text{gerbe } G \text{ on } M \quad \text{vector bundle } \mathcal{E}^{D_1}_{D_0} \]
  
  and a pair \((D_0, D_1)\) \[\mapsto\] on the space of \(G\)-branes of \(G\)-branes

  where \(IM = \{ \ell : [0, 1] \to M \mid \ell(0) \in D_0, \ell(1) \in D_1 \} \)

- The space of quantum states of the string spanning the branes \(D_0\) and \(D_1\) is

  \[ \mathcal{H}^{D_1}_{D_0} = \Gamma(\mathcal{E}^{D_1}_{D_0}) \quad \leftarrow \text{space of sections} \]

  with a geom. action of the affine algebra \(\hat{g}\) of level \(k\)
The spectrum of irreps. $\hat{V}_\lambda$ of $\hat{g}$

$$\mathcal{H}_{D_0}^{D_1} = \bigoplus_{\lambda} M_{D_1}^{D_1} \otimes \hat{V}_\lambda$$

found from the highest weight subspaces $M_{D_1}^{D_1} \subset \Gamma(\mathcal{E}_{D_0}^{D_1})$

$\rightarrow$ partition functions $Z_{D_1}^{D_2}(\tau)$

boundary operator product

Orientifolds

- Let \( \kappa : M \to M \) be an involution s.t. \( \kappa^* H = -H \)
  Let \( \mathcal{G} \) be a gerbe on \( M \) of curvature \( H \) and \( \mathcal{G}^* \) its dual (with curvature \( -H \)). A Jandle structure is a triple \((\kappa, \iota, \eta)\) where \( \iota \) and \( \eta \) are isomorphisms

\[
\kappa^* \mathcal{G} \cong \mathcal{G}^*, \quad \iota^2 \cong \text{Id}
\]

(Schreiber-Schweigert-Waldorf 2005). It permits to define the holonomy of \( \phi : \Sigma \to M \) for non-orientable \( \Sigma \) and the Feynman amplitudes of non-orientable strings

- Jandle structures on gerbes \( \mathcal{G}^k \) over \( G = \tilde{G}/\mathbb{Z} \) are obstructed by \( H^3(\Gamma, U(1)) \) and classified by \( H^2(\Gamma, U(1)) \) with \( \Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z} \) where the generator of \( \mathbb{Z}_2 \) acts by inversion on \( \mathbb{Z} \) and \( U(1) \)

- \( H^p(\Gamma, U(1)) \) may be calculated from the Lyndon-Hochschild-Serre spectral sequence (K.G.-Schweigert-Suszek-Waldorf, in writing)
Conclusions

- (Line-bundle) gerbes represent the background Kalb-Ramond fields in a non-trivial topology
- Gerbe modules represent twisted Chan-Paton gauge fields on branes to which couple the ends of open strings
- The classification of such structures on Lie groups leads to the classification of the WZW models and facilitates their solution on the classical and the quantum level
- It explains the origin and permits an explicit calculation of the finite group cohomology entering the solution
- Open problem: the dynamics of gerbes and their modules and its relations to the renormalization group flows