Clustering & collisions of particles suspended in turbulent flows

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Financial support by VR & by research platform ``Nanoparticles in an interactive environment” at GU
Small water drops in turbulent clouds

Stokes law

\[ \ddot{r} = \gamma (u(r, t) - \dot{r}) \]

\( u(r, t) \) turbulent flow (strength \( u_0 \)),

\[ \gamma = \frac{9 \rho_{\text{air}} \nu}{2 \rho_{\text{water}} a^2} \] Stokes constant,

\( \rho_{\text{air}} \) and \( \rho_{\text{water}} \) densities of air and water,

\( \nu \) kinematic viscosity,

and \( a \) particle size.

Figure 1 A turbulent cloud, illustrating the great range of spatial scales and the sharp boundaries characteristic of such a system. Similarly, a series of pictures taken in succession would illustrate the large range of temporal scales.
Model

Assumptions

1. spherical particles of mass $m$ and radius $a$ move independently (until they come into contact),
2. particles affect neither flow nor each other,
3. drag force given by Stokes law $\ddot{r} = \gamma (u(r, t) - \dot{r})$,
4. velocity field $u(r, t)$ isotropic, homogenous, stationary, and incompressible Gaussian random function.

Dimensionless parameters

$$St = \frac{1}{\gamma \tau}, \quad Ku = \frac{u_0 \tau}{\ell}, \quad n \ell^d, \quad a/\ell$$

$St$ Stokes number, $Ku$ Kubo number, $n$ particle density, $\ell$ and $\tau$ Kolmogorov length and time,
Mixing by random stirring

Computer simulation of $10^4$ particles (red) in two-dimensional random flow (periodic boundary conditions in space)

**a** initial distribution, **b** particle positions after random stirring.
`Unmixing´

Computer simulation of $10^4$ particles (blue) in two-dimensional smooth random incompressible flow $\mathbf{u}(r, t)$ (periodic boundary conditions in space)
Inertial particles in turbulent flow (statistical model, two-dimensional)

Particles floating on turbulent flow

Cressmann, Schumacher & Goldburg (2003)
Particles falling under gravity in turbulent flow

Lycopodium particles in turbulent channel flow

Fessler, Kulick & Eaton, Phys. Fluids 6 (1994) 3742
An example: correlated random walks

Consider $N$ random walks $x_i(t)$ ($i = 1, \ldots, N$), discrete in time ($t_n = n \delta t$)

$$x_i(t_{n+1}) = x_i(t_n) + \xi(x_i(t_n), t_n)$$

with Gaussian random displacements $\xi(x(t), t)$ satisfying

$$\langle \xi(x, t_n) \rangle = 0$$

$$\langle \xi(x, t_n) \xi(y, t_m) \rangle = \delta_{nm} \xi_0^2 e^{-(x-y)^2/2\ell^2}$$
An example: correlated random walks

Consider two particles with small initial separation $\delta x(0)$. Does $\delta x(t)$ typically decrease as $t \to \infty$?

Linearise

$$
\delta x(t_{n+1}) = \delta x(t_n) \left(1 + \frac{\partial \xi}{\partial x}(x(t_n), t_n)\right)
$$

and determine

$$
\lambda = \lim_{t \to \infty} \frac{1}{t} \left\langle \log \left| \frac{\delta x(t)}{\delta x(0)} \right| \right\rangle
$$

$$
= \frac{1}{\delta t} \left\langle \log \left| 1 + \frac{\partial \xi}{\partial x} \right| \right\rangle
$$
An example: correlated random walks

\[ \lambda = \lim_{t \to \infty} \frac{1}{t} \left\langle \log \left| \frac{\delta x(t)}{\delta x(0)} \right| \right\rangle = \frac{1}{\delta t} \left\langle \log \left| 1 + \frac{\partial \xi}{\partial x} \right| \right\rangle \]

Assume that \( \partial \xi / \partial x \) is small (\( \xi_0 \ll \ell \)).

Neglect \( \cdots \), expand logarithm and average

\[ \lambda \approx -\frac{1}{2\delta t} \frac{\xi_0^2}{\ell^2} < 0 \]
An example: correlated random walks

\[ \lambda = \lim_{t \to \infty} \frac{1}{t} \left\langle \log \left| \frac{\delta x(t)}{\delta x(0)} \right| \right\rangle = \frac{1}{\delta t} \left\langle \log \left| 1 + \frac{\partial \xi}{\partial x} \right| \right\rangle \]

Assume that \( \frac{\partial \xi}{\partial x} \) is small (\( \xi_0 \ll \ell \)). Neglect \( \cdots \), expand logarithm and average

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Lyapunov exponents

Exponents $\lambda_1 > \lambda_2 > \lambda_3$ describe rate of contraction or expansion of small length element $\delta r_t$, area element $\delta A_t$, and volume element $\delta V_t$

$$
\lambda_1 = \lim_{t \to \infty} t^{-1} \log_e (\delta r_t)
$$

$$
\lambda_1 + \lambda_2 = \lim_{t \to \infty} t^{-1} \log_e (\delta A_t)
$$

$$
\lambda_1 + \lambda_2 + \lambda_3 = \lim_{t \to \infty} t^{-1} \log_e (\delta V_t)
$$

Stochastic differential equation

To calculate $\lambda_1 > \lambda_2 > \lambda_3$ express spatial separations $\delta r_\mu$ ($\mu = 1, 2, 3$) in terms of a co-moving coordinate system $n_\mu(t) = O(t)n_\mu(0)$, momentum separations $\delta p_\mu$ as $\delta p_\mu = R \delta r_\mu$. Find

$$\lambda_\mu = \langle R'_{\mu\mu} \rangle / m$$

$$\dot{R}' = -\gamma R' - R'^2 / m + [R', O^\dagger \dot{O}]_- + F'$$

$$F_{\mu\nu}(t) = \gamma m \frac{\partial u_\mu}{\partial r_\nu}$$

and

$$R'_{\mu\nu}(t) = n_\mu(t) \cdot R(t)n_\nu(t)$$

as well as

$$F'_{\mu\nu}(t) = n_\mu(t) \cdot F(t)n_\nu(t)$$

For rapidly fluctuating and weak forcing obtain generalised diffusion equation for $R'$ which can be mapped onto a quantum problem.
Mapping onto quantum problem

Generalised diffusion equation for $3 \times 3$ matrix $\mathbf{R}'$ equivalent to perturbation of nine-dimensional isotropic harmonic oscillator

$$\hat{H} = \hat{H}_0 + \mathcal{I}^{1/2} \hat{H}_1$$

$$\hat{H}_0 = -\sum_{i=1}^{9} \hat{a}_i^\dagger \hat{a}_i$$

$$\hat{H}_1 = -\sum_{i,j,k} H^{(1)}_{i,j,k} \hat{a}_i^\dagger (\hat{a}_j^\dagger + \hat{a}_j) (\hat{a}_k^\dagger + \hat{a}_k)$$

where $\mathcal{I} = \frac{1}{2\gamma} \int_{-\infty}^{\infty} dt \langle \frac{\partial u_1}{\partial x_1}(\mathbf{r}(t), t) \frac{\partial u_1}{\partial x_1}(\mathbf{r}(0), 0) \rangle \propto K u^2 St$ is dimensionless measure of turbulence intensity.

Coefficients $H^{(1)}_{i,j,k}$ exactly known. Lyapunov exponents are obtained as matrix elements between ground state of $\hat{H}_0$ and the state $|Q\rangle$ given by $\hat{H} |Q\rangle = 0$. 
Algebraic perturbation theory

\[ \lambda_1 / \gamma = 3I - 29I^2 + 564I^3 - 14977I^4 + 488784I^5 - 18670570I^6 + \cdots \]

\[ \lambda_2 / \gamma = 8I^2 - 459/2 I^3 + 14281/2 I^4 - 757273/3 I^5 + 361653709/36 I^6 + \cdots \]

\[ \lambda_3 / \gamma = -3I - 9I^2 - 789/2 I^3 - 5787/2 I^4 - 895169/3 I^5 - 101637719/36 I^6 + \cdots . \]


As \( I \to 0 \) obtain known results for advective limit (\( \lambda_1 + \lambda_2 + \lambda_3 = 0 \))

Falkovich, Gawedzki & Vergassola, Rev. Mod. Phys. 73 (2001) 913
Perturbation series in one dimension

Obtain series expansion for \( \lambda_1 \)

\[
\lambda_1 / \gamma = - \sum_{l=1}^{\infty} c_l \mathcal{I}^l.
\]

Coefficients satisfy recursion ( \( c_1 = 1 \) )

\[
c_{l+1} = (6l - 2) c_l + \sum_{j=1}^{l} c_j c_{l+1-j}.
\]

Same coefficients appear in

Algorithmica 22 (1998) 490
Asymptotic series

Divergente rækker er i det Hele noget Fandenskap og det er en skam at man vover at grunde nagon Demonstration derpaa.

N. H. Abel (1828)
Result of resummation

Maximal Lyapunov exponent $\lambda_1$ for incompressible flow, $\Gamma = 2$. Partially compressible, $\lambda_1$ and $\lambda_2$. For $\lambda_2$, contribution $e^{-1/(6I)}$ is missing.

Fractal clustering

Dimension deficit $\Delta$. For particles in $d = 3$ incompressible flow:

$$\Delta = -\frac{1}{|\lambda_3|} (\lambda_1 + \lambda_2 + \lambda_3) .$$

Fractal dimension of attractor

$$d_f = d - \Delta \quad (\text{when } \Delta > 0).$$

From Pade-Borel resummation of series for Lyapunov exponents obtain good agreement with direct numerical simulations of particles suspended in turbulent flow (Bec et al. nlin.CD/0606024).

Since $Ku$ is not known for turbulent flow, adjusted $x$-axis by setting $Ku = 0.25$.

Caustics

Infinityesimal volume element $\delta V_t$ contracts or expands on average. But nothing prevents it from collapsing to zero for an instant of time: singularity in particle density $\rho$.

Consider one spatial dimension: $\delta x_t = 0$ corresponds to singularity in $X = \delta v/\delta x$.
Caustics in two dimensions

Density of particles suspended in a random flow (compressible, $\mathbf{u} = -\nabla \phi$)

Caustics of sun light in water

http://www.physics.utoronto.ca/~peet/
Caustic activation

One spatial dimension. Consider small separation in space $\delta x$ and in velocity $\delta v$. Equation of motion for $R = m \delta v/\delta x$

$$\dot{R} = -\gamma R - R^2/m + m\gamma \frac{\partial u}{\partial x} (x(t), t)$$

Stochastic driving $\partial u/\partial x$. When fluctuations drive $R$ sufficiently negative, it will almost certainly escape to $-\infty$ in a finite time and return from $\infty$: caustic.

Analogous to Kramers escape, obtain Arrhenius law for rate $J$ of formation of caustics (with action $S$)

$$\frac{J}{\gamma} \sim e^{-S/I}$$ with $$I \propto Ku^2 St$$
Collisions

Low density \( (n\ell^d \ll 1) \). Integrate \( \ddot{\mathbf{r}} = \gamma (\mathbf{u}(\mathbf{r},t) - \dot{\mathbf{r}}) \), ask: at which rate \( R \) would particles collide if they were hard spheres of radius \( a \)?

If particles were advected by the flow \( \mathbf{u}(\mathbf{r},t) \), local shear would give rise to collisions. Assuming time-independent shear obtain \( (\text{Saffmann & Turner, J. Fluid Mech. 1 (1956) 16}) \)

\[
R_{\text{adv}} \propto na^d u_0 / \ell
\]

Because of recollisions and rapid shear fluctuations in time this is at best an upper bound. Yet it gives collision rates in clouds too small by many orders of magnitude.
Collision rate

Consensus: turbulence increases collision rate through fluctuations in the particle density (preferential concentration).

But consider caustics. Two effects: fluctuations in particle density and increase in relative velocity.

Relative velocity randomised (because Lyapunov exponent $\lambda$ large): collision rate $R$ given by Boltzmann kinetics when caustics are frequent

$$R = R_{\text{adv}} + \exp(-S/I)R_{\text{kin}}$$

Saffmann & Turner, J. Fluid Mech. 1 (1956) 16


Saffmann & Turner, J. Fluid Mech. 1 (1956) 16
Results for collision rate

where $\mathcal{I} \propto K u^2 S t$.
Estimate for clouds

Precipitous increase of collision rate $R$ when $\mathcal{I}$ exceeds $S$.

Size of water droplets $a \sim 10\mu$m, density $n \approx 10^8\text{m}^{-3}$

Stokes damping $\gamma \approx 500\text{s}^{-1}$

Kolmogorov length $\ell \approx 3 \times 10^{-4}\text{m}$ and time $\tau \approx 10^{-2}\text{s}$

Obtain $R \approx 5 \times 10^{-3}\text{s}^{-1}$ if $\mathcal{I}$ exceeds $S$. Conclude: rainfall can be initiated in a timescale of a few minutes, provided that a sufficiently large part of the cloud has $\mathcal{I} > S$. 
Caustic purge

A summer afternoon: the air is hot, and a flock of cumulus clouds hover in the blue sky. Suddenly it pours down. Such an abrupt onset of rainfall from these clouds might be due to the formation of so-called fold caustics in the velocity field of the raindrops, report Michael Wilkinson and colleagues (Phys. Rev. Lett. in the press; http://arxiv.org/cond-mat/0604166).

It has been accepted for some time that small-scale turbulence, typical in cumulus clouds, is involved in the process of initiating rain showers. But most studies have assumed that cluster formation might be the relevant mechanism. Wilkinson et al. follow a different path. They argue that when the intensity of the turbulence increases, at some point fast droplets suddenly start to overtake slower ones. Then, at certain locations inside the cloud, the velocity field of the droplets takes several values—a caustic forms. This relative motion between the droplets could produce a sudden increase in collision rate, resembling an activation process, where the intensity of the turbulence plays the role of temperature.

And once the intensity passes a certain threshold, you get wet.

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Downpours Demystified?

By Adrian Cho
ScienceNOW Daily News
14 July 2006

As anyone who's been caught without an umbrella knows, even a fluffy, innocuous-looking cloud can unleash a sudden torrent of rain. A new theory may explain why.

Raindrops form as micrometer-sized droplets of moisture in a cloud collide and merge. Although researchers can reproduce this process in computer simulations, they aren't sure why the droplets merge. Many theories focus on clustering of droplets, but those theories face some fundamental problems in explaining the sudden onset of rain, says Bernhard Mehlig, a physicist at the Göteborg University in Sweden. For one thing, the density of droplets is so small—less than one droplet per cubic millimeter—that such droplets would not be able to form a cluster that could collide and merge into a raindrop quickly enough to cause a sudden rain shower.