Analysis of eigenvalue ensembles using Ward identities

Yacin Ameur
Main collaborators: N. G. Makarov, N.-G. Kang

Centre for Mathematical Sciences
Lund University, Sweden
Yacin.Ameur@maths.lth.se

Helsinki, April 6 2016
Particle systems

A system \( \{ \zeta_i \}_{i=1}^n \in \mathbb{C} \) ("point charges") in external field \( nQ \).

- **Energy:**
  \[
  H_n = \sum_{j \neq k}^{n} \log \frac{1}{|\zeta_j - \zeta_k|} + n \sum_{j=1}^{n} Q(\zeta_j).
  \]

- **Boltzmann–Gibbs law:**
  \[
  dP_n(\zeta) = \frac{1}{Z_n^\beta} e^{-\beta H_n(\zeta)} d^{2n} \zeta, \quad \zeta = (\zeta_j)_{1}^{n}.
  \] (1)

- **Assumptions.** \( Q : \mathbb{C} \rightarrow \mathbb{R} \cup \{+\infty\} \) is l.s.c., \( C^\omega \)-smooth, and
  \[
  Q(\zeta) \gg \log |\zeta|, \quad (\zeta \rightarrow \infty).
  \]

A minimizer \( \{ \zeta_j \}_{1}^{n} \) of \( H_n \) is a **Fekete-configuration**.
Frostman’s equilibrium measure

- **Q-energy** of a Borel p.m. $\mu$ on $\mathbb{C}$

$$I(\mu) := \iint \log \frac{1}{|\zeta - \eta|} \, d\mu(\zeta)d\mu(\eta) + \int Q \, d\mu.$$ 

The **equilibrium measure** $\sigma$ minimizes $I(\mu)$ where $\mu(\mathbb{C}) = 1$.

- **Droplet**

$$S = S[Q] := \text{supp } \sigma. \quad (2)$$

**Theorem**

(Frostman)

$$d\sigma(z) = \chi_S(z) \, \Delta Q(z) \, dA(z).$$

(In particular $\Delta Q \geq 0$ on $S$.)

**Conformal metric:**

$$ds^2 = \Delta Q(\zeta)|d\zeta|^2.$$

Abrikosov: Fekete configuration $\leftrightarrow$ honeycomb lattice.
Nature of droplets

Lemma
Fix \( p \in \partial S \). There is a nbh \( N \) of \( p \) and a "local Schwarz function" \( s(\zeta) \) on \( N \) obeying
- \( s \) is analytic in \( N \setminus S \),
- \( s \) is continuous on \( N \) and

\[
\bar{\zeta} = s(\zeta), \quad \zeta \in (\partial S) \cap N.
\]

Theorem
(Sakai, 1991) \( \partial S \) is a union of finitely many analytic curves. Possible singularities: cusps pointing out of \( S \) and double points.

Complement \( S^c \) is an Unbounded Quadrature Domain (in wide sense of Shapiro).
Droplets 1

Technical assumptions:
- $Q$ is real-analytic in a nbh of $S$.
- $\Delta Q > 0$ in a nbh of $\partial S$.

**Figure:** The Deltoid is not admissible; it has three maximal 3/2 cusps. 5/2 cusp is OK.
Figure: Double point and 5/2 cusp under Hele-Shaw flow.
Joint intensities

Let \( \{\zeta_j\}_1^n \) random sample. \( k \)-point function

\[
R_{n,1}(\eta) = \lim_{\epsilon \to 0} \frac{P_n(D(\eta; \epsilon) \cap \{\zeta_j\}_1^n \neq \emptyset)}{\epsilon^2},
\]

\[
R_{n,2}(\eta_1, \eta_2) = \lim_{\epsilon \to 0} \frac{P_n(D(\eta_l; \epsilon) \cap \{\zeta_j\} \neq \emptyset, \quad l = 1, 2)}{\epsilon^4}, \quad \text{etc.}
\]

Asymptotics as \( n \to \infty \) should give CFT.

If \( \beta = 1 \), the process is \textit{determinantal},

\[
R_{n,k}(\eta_1, \ldots, \eta_k) = \det \left( K_n(\eta_i, \eta_j) \right)_{i,j=1}^k.
\]

Here \( K_n \) is a "correlation kernel" = reprokernel for

\[
\mathcal{N}_n := \{ q \cdot e^{-nQ/2}; \ \text{degree}(q) < n \} \subset L^2.
\]

Note: \( E_n(f(\zeta_1, \ldots, \zeta_k)) = \frac{(n-k)!}{n!} \int_{\mathbb{C}^k} f \cdot R_{n,k} \).
"Classical" convergence result (for all $\beta$)

Random measure

$$\mu_n := \frac{1}{n} \sum_{1}^{n} \delta_{\zeta_j}.$$  

For $f \in W^{1,2}$

$$\sigma_n(f) := E_n(\mu_n(f)) = \frac{1}{n} \sum_{1}^{n} E_n(f(\zeta_j)) = \frac{1}{n} \int_{\mathbb{C}} f \cdot R_n.$$  

Theorem

$$(MH) \frac{1}{n} R_n \, dA \to \sigma \text{ and } $$

$$\sigma_n(f) \to \int f \, d\sigma, \quad (n \to \infty).$$

(Here $R_n = R_{n,1}$.)
Example: Ginibre ensemble ($\beta = 1$)

Let $Q(\zeta) = |\zeta|^2$. Then $S = \{ |\zeta| \leq 1 \}$ and $\sigma = \chi_S \, dA$.

The process $\{\zeta_i\}_1^n$ can be interpreted as eigenvalues of an $n \times n$-matrix with i.i.d. centered complex Gaussian entries of variance $1/n$.

Figure: A sample of the Ginibre process for a large value of $n$. We will later look at the process near the boundary.
Fluctuation theorem ($\beta = 1$)

Random measures $\text{fluct}_n := n(\mu_n - \sigma) = \sum_1^n \delta\zeta_j - n\sigma$.

Random variables on $\left(\mathbb{C}^n, \mathbb{P}_n\right)$

$$\text{fluct}_n(f) = \sum_{1}^{n} f(\zeta_j) - n\sigma(f), \quad (f \in \mathcal{C}_b^\infty(\mathbb{C})).$$

**Theorem**

$\text{fluct}_n(f)$ converges in distribution to the normal $N(e_f, \sigma_f^2)$, where

$$e_f = \frac{1}{8\pi} \int_{\mathbb{C}} f \cdot \Delta(\chi_S + L^S), \quad \sigma_f^2 = \frac{1}{2} \int_{\mathbb{C}} |\nabla f^S|^2, \quad (L = \log \Delta Q).$$

Here $f^S$ equals $f$ in $S$ and is harmonic and bounded in $\mathbb{C} \setminus S$. 

Yacin Ameur Main collaborators: N. G. Makarov, N.-G. Kang (LU)

Analysis of eigenvalue ensembles using Ward identities

Helsinki, April 6 2016 10 / 42
There is no \((1/\sqrt{n})\)-normalization!

Theorem says that random distributions

\[
\text{fluct}_n - \Delta(\chi_S + L^S)
\]

converge to GFF on \(S\) with free boundary conditions.

Test-class should be \(f \in W^{1,2}(\mathbb{C})\): GFF is an isometry \(\Phi: W^{1,2}(\mathbb{C}) \rightarrow L^2(P)\) such that \(\Phi(f)\) is centered normal with variance \(\int |f^S|^2 \, dA\), and (Wick’s formalism)

\[
\langle \Phi(f_1) \cdots \Phi(f_{2p}) \rangle = \sum_{k=1}^p \prod_{k=1}^p \langle f_{i_k}, f_{j_k} \rangle \nabla.
\]

The theorem is only proved for connected \(S\) with everywhere regular boundary.

There are "physical" results for arbitrary \(\beta\); also results by Johansson in dim 1. Our method "should" extend, but we need some estimates.
Field approximations: $CFT_n$

Let $(\zeta_j)_1^n$ and $(\zeta'_j)_1^n$ independent random samples, and put

$$\Phi_n(z) = 2 \sum_{j=1}^{n} (G(z, \zeta_j) - G(z, \zeta'_j))$$

where $G$ is Green's function for $S$. In a sense $\Phi_n$ converges to the GFF $\Phi$ on $S$ with Dirichlet boundary conditions, for example ($\beta = 1$)

$$E_n [\Phi_n(z)\Phi_n(w)] = G(z, w) + o(1),$$
$$E_n \left[ \Phi_n(z)^2 \right] = \log \sqrt{n} + (1 + \gamma)/2 + c(z) + o(1),$$

where $c(z) = \lim_{w \to z} (G(z, w) + \log |z - w|)$ is log-conformal radius. For suitable "finite parts", $\Phi_n^2 \to \Phi^*^2$ (OPE square).

Similarly, one can obtain other constructions from CFT on a concrete level.
Ward’s identity

Let $\{\zeta_j\}_1^n$ system. For smooth $\psi$ define r.v.’s

$$A_\psi = \frac{1}{2} \sum_{j \neq k}^n \frac{\psi(\zeta_j) - \psi(\zeta_k)}{\zeta_j - \zeta_k}, \quad B_\psi = n \sum_1^n \partial Q(\zeta_j) \psi(\zeta_j), \quad C_\psi = \sum_1^n \partial \psi(\zeta_j).$$

**Theorem**

*For all* $\psi$

$$E_n(\beta \cdot (A_\psi - B_\psi) + C_\psi) = 0.$$

This is an implicit relation between $R_{n,1}$ and $R_{n,2}$.

(Proof: reparametrization invariance of the partition function

$$Z_n := \int_{\mathbb{C}^n} e^{-\beta H_n} dV_n.$$
Rescaling

\[ \{ \zeta_j \}^n_1 \text{ random sample from } P_n. \text{ Fix } p_n \in S, \theta_n \in \mathbb{R}. \]

**Mesoscopic scale:** \( r_n = r_n(p_n) \) satisfies:

\[
n \cdot \int_{D(p_n,r_n)} \Delta Q(\zeta) \, dA(\zeta) = 1.
\]

If \( \Delta Q(p) > 0 \) then \( r_n \sim 1/\sqrt{n\Delta Q(p)} \).

**Rescaled system:**

\[
z_j = e^{-i\theta_n r_n^{-1} (\zeta_j - p_n)}.
\]
Rescaled process \((\beta = 1)\)

Rescaled process \(\Theta_n := \{z_j\}_1^n\) has \(k\)-point function

\[
R_{n,k}(z_1, \ldots, z_k) = r_n^{2k} R_{n,k}(\zeta_1, \ldots, \zeta_k).
\]

We have:

\[
R_{n,k}(z_1, \ldots, z_k) = \det(K_n(z_i, z_j))_{i,j=1}^k, \quad K_n(z, w) := r_n^2 K_n(\zeta, \eta).
\]

**Known:** if \(p \in \text{Int } S\), then \(R_{n,k}(z_1, \ldots, z_k) \rightarrow \det(G(z_i, z_j))_{k \times k}\) where

\[
G(z, w) = e^{z \bar{w} - |z|^2/2 - |w|^2/2}.
\]

is Ginibre kernel.
Compactness and analyticity

A **cocycle** is a function $c(z, w) = g(z) \bar{g}(w)$ where $g$ is continuous and unimodular. Correlation kernels are only determined up to cocycles.

**Theorem**

There are cocycles $c_n$ such that (on subsequences)

$$c_{n_k} K_{n_k} \to K, \quad \text{where} \quad K(z, w) = G(z, w) \psi(z, w).$$

Here $G(z, w) = e^{-|z|^2/2-|w|^2/2+z\bar{w}}$ and $\psi(z, w)$ is Hermitian entire.

(Proof: Taylor’s formula + normal families.)

A limit point $K = G\psi$ is called a **limiting kernel** at (moving) point $p$. 
Limiting point fields

A limiting kernel $K$ is correlation kernel of a limiting point field $\{\zeta_j\}_{1}^{\infty}$ with $k$-point function $R_k(\eta_1, \ldots, \eta_k) = \det(K(\eta_i, \eta_j))_{i,j=1}^{k}$. Limiting 1-pt function

$$\begin{align*}
R(z) &:= K(z,z) = \Psi(z,z).
\end{align*}$$

$R$ determines $\Psi$ by polarization and $K = G\Psi$ so

$R$ determines all $k$-point functions.
Forrester–Honner’s formula

**Theorem**

Let $Q = |ζ|^2$ and rescale about a boundary point in the outer normal direction. Then $R(z) = F(z + \bar{z})$ where $F$ is "free boundary plasma function"

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-(z-t)^2/2} \, dt.$$  

**Figure:** $F$ is the analytic continuation to $\mathbb{C}$ of the d.f. of the standard normal.
Rescaled Ward identity

Suppose $R(z) = \psi(z, z) \neq 0$. Put

$$B(z, w) := \frac{|K(z, w)|^2}{R(z)} = e^{-|z-w|^2} \frac{|\psi(z, w)|^2}{\psi(z, z)}, \quad C(z) := \int \frac{B(z, w)}{z - w} dA(w).$$

**Theorem**

$R > 0$ everywhere, $C(z)$ is smooth, and we have Ward's equation

$$\bar{\partial} C(z) = R(z) - 1 - \Delta \log R(z).$$

Since $R \mapsto \psi$ by polarization, this is an equation for $R$!

**Note:** Ward's equation holds at any (moving) point s.t. $R \neq 0$. To fix $R$ uniquely, we need side-conditions. These depend on the nature of the point we're zooming on (bulk point, regular boundary pt, singular boundary pt).
Apriori estimates (side conditions)

Rescale about a regular boundary point in outer normal direction. Let $R(z)$ a limiting 1-point function.

1. Exterior estimate:

$$R(z) \leq Ce^{-2x^2}, \quad (x \geq 0).$$

2. 1/8-formula:

$$\int_{-\infty}^{+\infty} t \cdot (R(t) - \chi_{(-\infty,0)}(t)) \, dt = \frac{1}{8}.$$

(Proof: (1) by potential theory; (2) fluctuation theorem.)
Complementarity

A limiting kernel $K = G\Psi$ is a positive matrix in Aronszajn’s sense,

$$\sum_{j,k=1}^{N} \alpha_j \bar{\alpha}_k K(z_j, z_k) \geq 0.$$  

(Because $\det(K(z_j, z_k))_{N \times N} = R_N(z_1, \ldots, z_N) \geq 0$.)

**Theorem**

The complementary kernel

$$\tilde{K}(z, w) = G(z, w)(1 - \Psi(z, w))$$

is also a positive matrix. In particular $R(z) = \Psi(z, z) \leq 1$.

Warning: $\tilde{K}$ does not solve Ward, in general.
Translation Invariance (T.I.)

$R(z) = \psi(z, z)$ is called t.i. if $\psi(z + it, w + it) = \psi(z, w)$, $t \in \mathbb{R}$. Equivalently,

$$\psi(z, w) = \Phi(z + \bar{w})$$

where $\Phi$ is entire.
Gaussian representation

Theorem

If $K(z, w) = G(z, w)\Phi(z + \bar{w})$ is a t.i. limiting kernel, then there exists a Borel function $f$ on $\mathbb{R}$ with $0 \leq f \leq 1$ such that

$$\Phi(z) = \gamma * f(z) = \int_{-\infty}^{+\infty} \gamma(z - t)f(t) \, dt,$$

where $\gamma$ is the Gaussian kernel

$$\gamma(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

Examples:

- Bulk: $\Phi \equiv 1 = \gamma * \chi_{\mathbb{R}}$
- The plasma kernel: $F = \gamma * \chi_{(-\infty,0)}$

Proof: Uses Bochner’s theorem on positive definite functions (also for the complementary kernel).
T.i. solutions to Ward’s equation

**Theorem**

Let \( R(z) = \Phi(z + \bar{z}) \) a t.i. limiting 1-point function with \( R \not\equiv 0 \). Then \( R \) solves Ward’s equation iff there is an interval \( I \subset \mathbb{R} \) such that

\[
\Phi = \gamma * \chi_I.
\]

**Corollary**

If \( R(z) = \Phi(z + \bar{z}) \) is rescaled about a regular boundary point and \( R \) is t.i. then

\[
\Phi = F = \gamma * \chi_{(-\infty,0)}.
\]

**Comments:**

1. **Conjecture:** An arbitrary limiting kernel is t.i (exception: bulk singularities, degenerate boundary singularities).
2. In the general (non-t.i.) case, Ward is a twisted convolution equation. The physically relevant ones "should" be the above.
Consequences

For radial $Q(\zeta) = Q(|\zeta|)$ and for "ellipse potential" $Q = |\zeta|^2 - t\text{Re}(\zeta^2)$ we know that any limiting 1-point function $R$ is t.i.

Corollary

If $Q$ is one of the types above, rescale about a boundary point. The rescaled systems $\{z_j\}_1^n$ converges to the field with kernel

$$K(z, w) = G(z, w) \cdot F(z + \bar{w}).$$

- This should be true for a general potential at a regular boundary point.
- Lee and Riser obtained this for the ellipse potential $Q(\zeta) = |\zeta|^2 - t\text{Re}(\zeta^2)$ using orthogonal polynomials.
Singular boundary points

Now assume that $p$ is a cusp or a double point. If $p$ is a cusp, we assume it has type $(\nu, 2)$ where $\nu > 3$, i.e. it resembles

$$x^{\nu} = y^2.$$

**Theorem**

Rescale according to $z_j = \sqrt{n\Delta Q(p)}(\zeta_j - p)$. Then any limiting kernel is trivial: $R = 0$.

(Proof: exterior estimate (suitable coord system)

$$R(z) = \psi(z, z) \leq Ce^{-2x^2}.$$

Since $L(z, w) = e^{z\bar{w}}\psi(z, w)$ is Hermitian-entire and positive definite, log $L(z, z)$ is subharmonic. This gives $\psi = 0$, by the maximum principle.)
Cusps; moving points

Assume $S$ has a $(\nu, 2)$ cusp at $p$ and fix $T > 0$. Let $p_n \in S$ be the point of distance $\frac{T}{\sqrt{n\Delta Q(p)}}$ from the boundary which is closest to $p$.

Rescale about $p_n$

$$z_j = e^{-i\theta_n}r_n^{-2}(\zeta_j - p_n), \quad j = 1, \ldots, n$$

where $\theta_n$ is chosen so that $e^{-i\theta_n}(p - p_n)$ is positive imaginary.
Existence theorem

**Theorem**

If $T$ is sufficiently large, then each limiting 1-point function $R(z) = K(z, z)$ is positive, satisfies Ward’s equation, and the estimate

$$R(z) \leq Ce^{-2(|x|-T)^2}.$$  \hfill (3)

- Estimate (3) shows that $R$ is associated with a "new" determinantal point field.
- After the rescaling, the droplet looks like the strip

$$\Sigma_T : \quad -T \leq \Re z \leq T,$$

so it is natural to assume that the field is t.i.
For $s > 0$ let

$$\Phi_s(z) = \gamma \ast \chi_{(-s,s)}(z) = \frac{1}{\sqrt{2\pi}} \int_{-s}^{s} e^{-\frac{(z-t)^2}{2}} \, dt.$$ 

If $s \leq 2T$ then $R_s(z) = \Phi(z + \bar{z})$ satisfies $R_s(x) \leq Ce^{-2(|x|-T)^2}$ and Ward’s equation.

- How should we choose $s$?
- In regular case, we used $1/8$-formula:

$$\int_{\mathbb{R}} t \cdot (R(t) - \chi_{(-\infty,0)}(t)) \, dt = \frac{1}{8}.$$

Something similar should hold at cusps.

**Conclusion.** We must extend the boundary fluctuation theorem to domains with cusps.
Natural candidates 2

Figure: The graphs of $R_T(x) := \Phi_{T/2}(2x)$ for $T = 2, 5, 8$. 
The hard edge (Neumann B.C.’s)

Let \( Q \) be a potential. Define \( Q^S = Q \) on \( S \) and \( Q^S = +\infty \) otherwise. Let \( \{\zeta_j\} \) be a random sample from \( P_n \) and rescale about a regular boundary point \( p \) to obtain \( \{z_j\}_1^n \).

**Theorem**

For u.t.i. potentials, the processes \( \{z_j\}_1^n \) converge to a unique point field with correlation kernel

\[
K(z, w) = G(z, w)H(z + \bar{w})\chi_L(z)\chi_L(w), \quad L = \{Re z < 0\},
\]

where \( H \) is the hard edge plasma function,

\[
H(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \frac{e^{-(z-t)^2/2}}{F(t)} dt,
\]

(\( F = \)free boundary plasma function).
The hard-edge theory is parallel to the free boundary; we can obtain existence of new hard edge fields near cusps, and so on.

**Remaining problem.** We need a fluctuation theorem for the hard edge to completely characterize the limiting kernels. (We can now do it up to a constant.)
Natural candidates 3: hard edge near a singular point

For $T > 0$ define

$$H_T(z) := \frac{1}{\sqrt{2\pi}} \int_{-2T}^{2T} \frac{e^{-(z-t)^2/2}}{F_T(t)} \, dt, \quad (F_T = \gamma \ast \chi(-2T,2T)).$$

The "1-point function" is then $R^h_T(z) := H_T(z + \bar{z})\chi(-T,T)(\text{Re } z)$.

Figure: The graph of $R^h_T$ restricted to the reals, for $T = 2$, $T = 5$, and $T = 8$. 
Suppose that $0 \in \text{Int } S$ and $\Delta Q(0) = 0$. Taylor expansion

$$Q(x + iy) = Q_0(x + iy) + \Re H(z) + \ldots$$

where $Q_0(x + iy) = \sum_{j=0}^{2k} a_j x^j y^{2k-j}$ is homogeneous of degree $2k$. Mesoscopic scale $r_n$ satisfies $n\sigma(D(0, r_n)) = 1$, i.e.

$$r_n \sim n^{-1/2k}.$$ 

Limiting rescaled kernels take the form

$$K(z, w) = L(z, w) e^{-Q_0(\tau_0 z)/2 - Q_0(\tau_0 w)/2}.$$

Let $L_0(z, w)$ be the Bergman kernel of $L^2_{\mu_0}$, $d\mu_0 = e^{-Q_0(\tau_0 \cdot)} dA$ and $R_0(z) = L_0(z, z)e^{-Q_0(\tau_0 z)}$. 

Bulk Singularities 2

Under suitable assumptions on $Q$ we have $R = R_0$ for each limiting 1-point function. If $Q_0 = |\zeta|^{2k}$, $R_0(z) = M_k(|z|^2)e^{-Q_0(z)}$ where $M_k$ is a Mittag-Leffler function.

Figure: $R_0$ for $Q_0 = |z|^4 - |z|^2\text{Re}(z^2)/2$ and the graph of the Mittag-Leffler function $M_2(x^2)$
Bulk singularities 3

Berezin kernels $B(z, w) = |L_0(z, w)|^2 e^{-Q_0(\tau_0 w)} / L_0(z, z)$ corresponding to $Q = |\zeta|^4$.

Figure: Berezin kernels rooted at 0 and at 1

Ward: $\bar{\partial}C = R - \Delta Q_0(\tau_0 \cdot) - \Delta \log R$. 
Logarithmic singularities

Potential $Q(\zeta) = c_1|\zeta|^{2\lambda} + 2c_2 n^{-1} \log |\zeta| + \ldots$ has mesoscopic scale $r_n$ satisfying $c_2 + c'_1 n r_n^{2\lambda} = 1$. Rescaling on the scale $r_n$ we find that limiting 1-point functions are of the type $R_{\lambda,\mu}(z) = cE_{\lambda,\mu}(|z|^2)e^{-Q_0(\tau_0 z)}$ where

$$Q_0(z) = |z|^{2\lambda} + 2(1 - \lambda/\mu) \log |z|$$

and $E_{\lambda,\mu}(z) = \sum \frac{z^j}{\Gamma(\lambda_j + \mu)}$ a Mittag-Leffler function. The case $c_2 < 0$ corresponds to conical singularities, $c_2 > 0$ corresponds to branch-points on the Riemann surface associated with $Q$.

![Figure: The 1-point functions $R_{1,2}$ and $R_{2,4}$.](image)
Fekete points

Let $p$ be a regular boundary point, and let $\mathcal{F}_n = \{\zeta_{jn}\}_{j=1}^n$ be an $n$-Fekete configuration, $n = 1, 2, \ldots$. Denote

$$d_n(\zeta_{nj}) = \sqrt{n\Delta Q(\zeta_{nj}) \cdot \min_{k \neq j} |\zeta_{nj} - \zeta_{nk}|},$$

and ("asymptotic separation constant")

$$\Delta(\mathcal{F}) = \liminf_{n \to \infty} \min_{j=1,\ldots,n} \left\{ d_n(\zeta_{nj}) \right\}.$$

**Theorem**

$$\Delta(\mathcal{F}) \geq 1/\sqrt{e}.$$  

We believe that $\Delta(\mathcal{F}) = \sqrt{2/\sqrt{3}}$.  
(This comes close to Abrikosov's conjecture.)
Comparison: the one-dimensional case

If $Q$ is real-analytic on $\mathbb{R}$ and $Q = +\infty$ outside $\mathbb{R}$, then $S$ is a finite union of compact intervals. Let $p$ be a boundary point. Rescale by

$$x_j = cn^{2/3}(\xi_j - p).$$

Theorem

In free boundary case, there is a $c$ such that $\{x_j\}^n_1$ converges to the Airy process with kernel

$$K(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y}.$$ 

In hard edge case we get a Bessel process with kernel

$$K(x, y) = \frac{\sqrt{x}\sqrt{y} J_0'(\sqrt{y}) - \sqrt{x} J_0'(\sqrt{x}) J_0(\sqrt{y})}{2(x - y)}.$$ 

These are reprokernels for "de Branges spaces". The two-dimensional plasma kernels are quite different.
We don’t now have a kernel, but we put

\[ B(z, w) = \frac{R(z)R(w) - R_2(z, w)}{R(z)} \]

and

\[ C(z) = \int_C \frac{B(z, w)}{z - w} dA(w). \]

Ward’s equation is

\[ \bar{\partial}C = R - 1 - \frac{1}{\beta} \Delta \log R. \]

The equation needs to be "closed". When \( \beta = 1 \) we used the extra structure of existence of a kernel \( K = G\psi \).
In the "regular bulk case", Jancovici obtained the expansion
\[ B^\beta(z, w) = e^{-|z-w|^2} + (\beta - 1)f(|z - w|) + O((\beta - 1)^2) \] with a certain explicit \( f \).

At a regular boundary point, Wiegmann et al proves
\[ B^\beta(z, w) = e^{-|z-w|^2}|F(z + \bar{w})|^2/F(z + \bar{z}) + (\beta - 1)f(z, w) + \ldots. \]

Problems

1. What does Ward’s equation say about the correction \( f \)?
2. What is the correction in bulk singular case?
Degenerate boundary singularities

There can be points at the boundary where $\Delta Q = 0$. Example: $Q = |\zeta|^4 - \sqrt{2}\text{Re}(\zeta^2)$ gives a "figure 8" at 0.

1. Can we obtain new non-trivial point-fields by zooming at a suitable moving point approaching 0?
2. Can one characterize the degenerate boundary singularities a la Sakai?