Superfluidity and Quantum Metric

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Outline
Introduction

- Superconductivity
- Superfluid weight and superfluid mass
- Lattice systems: Bloch theorem, band structure and effective mass
- Flat bands enhance the superconducting critical temperature

Our work

- Main question: is there superfluid transport in a flat band?
- Mean field approach
- Perturbative approach
- Numerical approach: ED, DMFT, DMRG

Perspectives
References to our works


Superconductivity as a symmetry breaking

Superconductivity is characterized by a nonzero order parameter

$$\Delta(x) = -g \langle \hat{\psi}_\downarrow(x) \hat{\psi}_\uparrow(x) \rangle$$

Not gauge invariant, spontaneous breaking of U(1) gauge symmetry

The order parameter is frozen in the mean-field approximation (BCS theory)

$$-g \int dx \hat{\psi}_\uparrow^\dagger(x) \hat{\psi}_\downarrow^\dagger(x) \hat{\psi}_\downarrow(x) \hat{\psi}_\uparrow(x) \approx \int dx \left( \Delta(x) \hat{\psi}_\uparrow^\dagger(x) \hat{\psi}_\downarrow^\dagger(x) + \text{H.c.} \right)$$
Ginzburg-Landau equations

Ginzburg-Landau equations describe the order parameter (macroscopic wavefunction $\Psi \approx \Delta$) in nonhomogeneous and time-dependent cases.

\[
\frac{1}{2m} \left(-i\hbar \nabla - 2eA\right)^2 \Psi + \alpha \Psi + \beta |\Psi|^2 \Psi = 0
\]

\[
j = \frac{2e}{m} \text{Re} \left[ \Psi^* \left(-i\hbar \nabla - 2eA\right) \Psi \right] \quad \nabla \times B = \mu_0 j
\]

Coherence length $\xi = \sqrt{\frac{\hbar^2}{2m|\alpha|}}$

London penetration depth $\lambda = \sqrt{\frac{m}{4\mu_0 e^2 n_s}}$

Decay of order parameter (proximity effect)

Decay of magnetic fields and currents (Meissner effect)
Meissner effect and equilibrium supercurrent

The magnetic field is expelled by the superconductor bulk below the critical temperature.

The screening supercurrent flows on a surface layer with depth.

$$\lambda = \sqrt{\frac{m}{4\mu_0 e^2 n_s}}$$

The London penetration depth is related to the superfluid density by the formula (within GL eqs, valid only near Tc):

$$n_s = \frac{\Psi^2}{\Phi_0}$$
Question: what is the effect of the crystalline lattice structure on superconductivity?

Outstanding question which is still unanswered for High-Tc superconductors
Bloch theorem and band structure

Bloch theorem:
Diagonalization of the single-particle Hamiltonian with periodic potential leads to the band energy dispersions and the periodic Bloch functions.

Periodic potential \[ V(r) = V(r + a_i) \]

\[ H_{	ext{s.p.}} = -\frac{\hbar^2}{2m} \nabla^2 r + V(r) \]

\[ \begin{align*} \varepsilon_{nk} & \quad \text{band energies} \\
\gamma_{nk}(r) = \gamma_{nk}(r + a_i) & \quad \text{Bloch functions} \end{align*} \]

Definition of effective mass tensor

\[ \left( \frac{1}{m_{\text{eff}}} \right)_{ij} = \frac{1}{\hbar^2} \frac{\partial^2 \varepsilon_{nk}}{\partial k_i \partial k_j} \]
Wannier functions

Wannier functions form a localized and translationally invariant orthonormal basis

\[ w_n(r - r_i) = \frac{V_{\Omega}}{(2\pi)^3} \int d^3k e^{ik \cdot (r - r_i)} g_{nk}(r) \]

\[ = \frac{V_{\Omega}}{(2\pi)^3} \int d^3k e^{ik \cdot (r - r_i)} g_{nk}(r - r_i) \]

\[ i = (i_1, i_2, i_3) \]

\[ r_i = i_1 a_1 + i_2 a_2 + i_3 a_3 \]

In the basis of the Wannier functions of the lowest band a hopping Hamiltonian is obtained

\[ \hat{H} \approx \sum_{i,j,\sigma} K(i - j) \hat{c}_{i\sigma}^{\dagger} \hat{c}_{j\sigma} + \hat{H}_{\text{int}} \]
Simple lattices and composite lattices

**Simple lattices**
1 orbital per unit cell

**Composite lattices**
1 < orbitals per unit cell

Honeycomb lattice

Location of the orbitals

\[ \mathbf{r}_i = i_1 \mathbf{a}_1 + i_2 \mathbf{a}_2 + i_3 \mathbf{a}_3 \]

\(i, j\) label the unit cells

Location of the orbitals

\[ \mathbf{r}_{i\alpha} = i_1 \mathbf{a}_1 + i_2 \mathbf{a}_2 + i_3 \mathbf{a}_3 + \mathbf{b}_\alpha \]

\(\alpha, \beta\) label the sublattices (orbitals)
Effect of the lattice structure on the superconducting properties

The effective mass enters in the expression for the penetration depth

\[ \lambda = \sqrt{\frac{m}{4\mu_0 e^2 n_s}} \]

What if effective mass is infinite (FLAT BAND)?

Naively: Infinite penetration depth implies no Meissner effect, superconductivity is lost

Not quite so, a more meaningful quantity is the superfluid weight

\[ D_s = \frac{n_s}{m} \propto \lambda^{-2} \]
Mean-field ansatz for the superconducting state with finite ground state supercurrent

Order parameter is frozen and uniform in the simplest BCS ground state

\[ \Delta(r) = \Delta_0 \]

A state with uniform supercurrent corresponds to a plane-wave modulation of the order parameter phase

\[ \Delta(r) = \Delta_0 e^{2i\mathbf{q} \cdot \mathbf{r}} \]

Current flow in a superconductor is a ground state property!
Superfluid density and superfluid weight from the free energy

As a ground state property, superfluid density and superfluid weight can be related to the free energy change with the order parameter modulation.

Alternative definition of superfluid density and superfluid weight

\[ \frac{\Delta F}{V} = \frac{1}{2} \rho_s v_s^2 = \frac{1}{8} D_s p_s^2 \]

- \( v_s = \frac{\hbar q}{m} \) \quad Cooper pair velocity
- \( p_s = 2\hbar q \) \quad Cooper pair momentum
- \( J = \frac{1}{4} D_s \hbar q \) \quad Supercurrent density

It makes sense to define the **superfluid weight** even for a flat band (infinite mass).
Superfluid weight defines superconductivity

Nonzero pairing potential (binding energy of Cooper pairs) does not guarantee superconductivity

Example of a flat band composed of localized Wannier functions

A nonzero superfluid weight implies the Meissner effect and dissipationless transport
Why are flat band interesting?

Answer: high density of states implies high critical temperature according to BCS theory
Flat bands and room-temperature superconductivity

Dispersive band

$$ U \ll J \Rightarrow T_c \propto e^{-\frac{1}{Un_0(E_F)}} $$

Flat band

$$ U \gg J \Rightarrow T_c \propto Un_0(E_F) \propto U/J $$

Partial flat band of surface states in rhombohedral graphite

Kopnin, Heikkilä, Volovik, PRB 83, 220503(R) (2011)
Flat bands and high-temperature superconductivity

attractive Hubbard model in two dimensions with the following density of states (DOS):

1) uniform DOS

2) Square lattice, 2D van Hove singularity

Compare to the flat band case:

3) flat band DOS \( \rho(\varepsilon) = a\delta(\varepsilon) \)

<table>
<thead>
<tr>
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<th>U = 0.1 eV</th>
<th>U = 0.2 eV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform DOS</td>
<td>0.3 K</td>
<td>44 K</td>
</tr>
<tr>
<td>Square lattice DOS</td>
<td>24 K</td>
<td>183 K</td>
</tr>
<tr>
<td>Flat band DOS</td>
<td>124 K</td>
<td>290 K</td>
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</tbody>
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Bandwidth \( W = 1 \) eV
Main questions

A flat band is insulating for any filling in absence of interaction or disorder.

1) Can we characterize transport in a flat band once the interaction is turned on in terms of the properties of the noninteracting systems, i.e. from the Bloch functions?

2) Is there a relation between topological properties of the flat band and bulk transport properties?

3) What is the superconducting transition temperature in a flat band in the presence of interactions? What is the upper limit to the transition temperature?
The essential idea: multiband approach

**Question**
What properties of the flat band are associated to superfluid transport?

**Observation**
A flat band in a single-band lattice Hamiltonian is necessarily trivial since all the hopping matrix elements are vanishing. No transport can occur.

**Approach**
Use a multiband/multiorbital Hamiltonian defined on a composite lattice. The total Chern number must be zero and the interaction is to a good approximation a local Hubbard interaction.
Wannier functions in simple and composite lattices

\[ \mathbf{r}_{(1,1)3} = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{b}_3 \]

Unit cells (u.c.)
\[ N_c = 4 \]

Orbitals per u.c.
\[ N_\alpha = 3 \]

Position vector \( \mathbf{r}_{i\alpha} = i_\mathbf{a}_1 + i_\mathbf{b}_2 + b_\alpha \)

Chern number \( C_n \)
\[ C_2 \neq 0 \]
\[ \sum_n C_n = 0 \]

Composite band

Bloch plane wave
\[ \psi_{nk}(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} g_{nk}(\mathbf{r}) \]

Wannier functions
\[ w_n(\mathbf{r}) = \frac{V_\Omega}{(2\pi)^3} \int d^3k e^{i\mathbf{k} \cdot \mathbf{r}} g_{nk}(\mathbf{r}) \]

\[ w_\alpha(\mathbf{r}) = \frac{V_\Omega}{(2\pi)^3} \int d^3k e^{i\mathbf{k} \cdot \mathbf{r}} U_{\alpha,n}(\mathbf{k}) g_{nk}(\mathbf{r}) \]

The essential idea: multiband approach

The composite Wannier functions for a subset of bands are well-localized. One can use a simple Fermi-Hubbard tight-binding Hamiltonian.

The Wannier functions relative to the flat band are not necessary localized.
Why total Chern number = 0?

Exponential Localization of Wannier Functions in Insulators

Christian Brouder, Gianluca Panati, Matteo Calandra, Christophe Mourougane, and Nicola Marzari

Theorem:
Exponentially localized Wannier functions can be constructed in 2D or 3D iff the Chern number(s) is zero

\[ A(k) = i \langle g_{nk} | \partial_k g_{nk} \rangle \]  \hspace{1cm} \text{Berry connection}

\[ \Omega(k) = \hat{z} \cdot \nabla \times A(k) \]  \hspace{1cm} \text{Berry curvature}

\[ C = \frac{1}{2\pi} \int_{\text{B.Z.}} d^2k \Omega(k) \]  \hspace{1cm} \text{Chern number}

The zero Chern number of the composite band guarantees localized Wannier functions

The flat band can have nonzero Chern number
Several complementary approaches

- Mean-field BCS theory
- Perturbative approach
- Numerics (ED, DMFT, DMRG)
**Superfluid weight in a multiband/multiorbital system**

**Definition of superfluid weight**
\[ \frac{\Delta F}{V} = \frac{1}{2} \rho_s v_s^2 = \frac{1}{8} D_s \left( \hbar q \right)^2 \]

\[ D_s = D_{s,1} + D_{s,2} + D_{s,3} \]

\[ [D_{s,1}]_{i,j} = \frac{2}{V \hbar^2} \sum_k \text{Tr} \left[ V_k V_k^\dagger \partial_{k,i} \partial_{k,j} \varepsilon_k \right] \]

**Conventional contribution** present in the single band case

**Geometric contribution** present only in the multiband/multiorbital case

**Can be nonzero in a flat band**

\[ [G_k]_{\alpha,n} = g_{nk}(\alpha) \quad D_k(q) = -G^\dagger_{k-q} \Delta G_{k+q} \]

**Bloch functions**

\[ [D_{s,2}]_{i,j} = \frac{2}{V \hbar^2} \sum_k \text{Tr} \left[ V_k U_k^\dagger \partial_{q,i} \partial_{q,j} D_k \right] \]

\[ [D_{s,3}]_{i,j} = \frac{2}{V \hbar^2} \sum_k \sum_{n,n'} \frac{[B_{k,i}]_{n,n'} [B_{k,j}]_{n',n}}{E_{nk} + E_{n'k}} \]

\[ B_{k,i} = U_k^\dagger \partial_{q,i} D_k U_k + V_k^\dagger \partial_{q,i} D_k V_k + V_k^\dagger \partial_{k,i} \varepsilon_k U_k - U_k^\dagger \partial_{k,i} \varepsilon_k V_k \]

**Our result**

**Geometric contribution** present only in the multiband/multiorbital case

S. Peotta and PT, Nature Communications 6, 8944 (2015)
Flat band limit

Isolated flat-band approximation

bandwidth $W \ll U \ll E_g$ band gap

flat band isolated band

Uniform pairing assumption

$\Delta_\alpha = \Delta = \text{const.}$

$n^{-1}_\phi = N_\alpha$

Number of orbitals with nonzero pairing order parameter

partially filled flat band

$\nu$ flat band filling

$\Delta = U n_\phi \sqrt{\nu(1 - \nu)}$

$\left\{ \begin{array}{l}
C_2 \neq 0 \\
\sum_n C_n = 0
\end{array} \right.$

Composite band

S. Peotta and PT, Nature Communications 6, 8944 (2015)
Superfluid weight in a flat band

S. Peotta and PT, Nature Communications 6, 8944 (2015)

\[
[D_S]_{i,j} = \frac{4 n_\phi U \nu (1-\nu)}{(2\pi)^d \hbar^2} \int_{B.Z.} d^d k \, \Re B_{i,j}(k)
\]

Only the geometric contribution survives

Quantum Metric of the flat band
Superfluid weight in a flat band

S. Peotta and PT, Nature Communications 6, 8944 (2015)

Flat band Bloch functions: \( g_k(\alpha) = \langle \alpha | g_k \rangle \)

Bloch functions of the noninteracting lattice Hamiltonian

Quantum Geometric Tensor


\[
B_{ij}(k) = 2\langle \partial_{k_i} g_k | (1 - |g_k\rangle \langle g_k|) | \partial_{k_j} g_k \rangle
\]

Re \( B_{ij}(k) = g_{ij} \) Quantum metric

Im \( B_{ij}(k) \) Berry curvature

Distance between neighboring quantum states

\[
dl^2 = g_{ij} ds_i ds_j
\]
Riemannian Structure on Manifolds of Quantum States

J. P. Provost and G. Vallee
Physique Theorique, Universite de Nice


Distance between two nearby quantum states

\[ d\ell^2 = g_{ij} \, ds_i \, ds_j \]

Gauge invariant version:

\[ g_{ij}(s) = \gamma_{ij}(s) - \beta_i(s) \beta_j(s) \quad \beta_j(s) = -i(\psi(s), \partial_j \psi(s)) \]

\[ (\partial_i \psi, \partial_j \psi) = \gamma_{ij} + i\sigma_{ij} \]

Quantum metric

\[ g_{ij} + i\sigma_{ij} \]

Berry curvature

QGT
We show that:

1) Superfluidity in a flat band has a geometric origin (quantum metric)
Superfluid weight and Chern number

Superfluid weight in the isolated flat band limit

\[ [D_s]_{i,j} = \frac{4n_\phi U\nu(1-\nu)}{(2\pi)^d\hbar^2} \int_{B.Z.} d^d k \text{ Re } B_{i,j}(k) \]

\[ M_{ij} = \frac{1}{2\pi} \int_{B.Z.} d^2 k B_{i,j}(k) \]

Complex positive semidefinite matrix

\[ M_{ij}^R = \text{ Re } M_{ij} \]

\[ M_{ij} = M_{ij}^R + i\epsilon_{ij} C \geq 0 \implies \det(M^R) \geq |C|^2 \]

Chern number

Local form in \( k \)-space

\[ \det \text{ Re } B_{i,j}(k) \geq |\text{ Im } B_{i,j}(k)|^2 \]

\[ D_s \geq \int_{B.Z.} d^d k |\text{ Im } B_{i,j}(k)| \geq |C| \]

S. Peotta and PT, Nature Communications 6, 8944 (2015)
We show that:

1) Superfluidity in a flat band has a geometric origin (quantum metric)

2) The flat band superfluid weight is bounded from below by the Chern number
**Comparison between flat band and parabolic band**

**Parabolic band**

\[ D_s = \frac{n_p}{m_{\text{eff}}} \left(1 - \left(\frac{2\pi \Delta}{k_B T}\right)^{1/2} e^{-\Delta/(k_B T)}\right) \]

- Particle density \( n_p \)
- Bandwidth \( \frac{1}{m_{\text{eff}}} \propto J \)

Physical picture: global shift of the Fermi sphere

**Flat band**

\[ [D_s]_{i,j} = \frac{2}{\pi \hbar^2} \frac{\Delta^2}{U n_{\phi}} M_{i,j}^R \]

Linearly proportional to the coupling constant \( U \)!

Fingerprint of the geometric origin

**Physical picture:** delocalization of Wannier functions.

**Overlapping Cooper pairs:** pairing fluctuations support transport if pairs can be created and destroyed at distinct locations


Bounds on supersolids related to (dis)connectedness of the density
Extensions-Superfluid weight from linear response


Superfluid weight can be obtained from the current-current response
G. D. Baym, Mathematical methods in solid state and superfluid theory (1968)

\[ K_{\mu\nu}(q, \omega) = \langle T_{\mu\nu} \rangle - i \int_0^\infty dt e^{i(\omega+i0^+)t} \langle [j^p_{\mu}(q, t), j^p_{\nu}(-q, 0)] \rangle \]

\[ D^s_{\mu\nu} = K_{\mu\nu}(q \to 0, \omega = 0) \]

Consider the attractive Hubbard model on a composite lattice

\[ H = -\sum_{i\alpha, j\beta, \sigma} t^\sigma_{i\alpha, j\beta} c^\dagger_{i\alpha\sigma} c^\sigma_{j\beta\sigma} - U \sum_{i\alpha} n_{i\alpha\uparrow} n_{i\alpha\downarrow} - \mu \sum_{i\alpha\sigma} n_{i\alpha\sigma} \quad \text{BCS} \downarrow \]

\[ i : \text{unit cell label} \]
\[ \alpha = 1, 2, \cdots, M \text{ denotes the sublattice} \]

\[ H_{\text{MF}} = \sum_k \Psi_k^\dagger \mathcal{H}(k) \Psi_k \quad \mathcal{H}(k) = \begin{pmatrix} \mathcal{H}^\uparrow(k) - \mu & \Delta \\ \Delta^\dagger & -\mathcal{H}^\downarrow(-k) + \mu \end{pmatrix} \]

Nambu spinor \( \Psi_k = (c_{\alpha k\uparrow}, c^\dagger_{\alpha-k\uparrow})^T \) \( \Delta = \text{diag}(\Delta_1, \Delta_2, \cdots, \Delta_M) \) momentum independent

\[ T_{\mu\nu} = \sum_{k, \sigma} c^\dagger_{k\sigma} \partial_{\mu} \partial_{\nu} \mathcal{H}_\sigma(k) c_{k\sigma}, \]

\[ j^p_{\mu}(q) = \sum_{k, \sigma} c^\dagger_{k\sigma} \partial_{\mu} \mathcal{H}_\sigma(k+q/2) c_{k+q\sigma} \]
Extensions-Superfluid weight from linear response


\[ D_{\mu \nu}^s = \sum_{\mathbf{k}, i, j} \frac{n_f(E_j) - n_f(E_i)}{E_i - E_j} \left( \langle \psi_i | \partial_\mu \mathcal{H} | \psi_j \rangle \langle \psi_j | \partial_\nu \mathcal{H} | \psi_i \rangle - \langle \psi_i | \partial_\mu \mathcal{H} \sigma^3 | \psi_j \rangle \langle \psi_j | \sigma^3 \partial_\nu \mathcal{H} | \psi_i \rangle \right) \]

\( n_f \): Fermi – Dirac distribution. \( E \) and \( |\psi\rangle \): eigenvalue and eigenvector of \( \mathcal{H} \)

Noninteracting Bloch state

\[ |\psi_i \rangle = \sum_{m=1}^{M} \left( w_{+, im} |m\rangle \uparrow \otimes |+\rangle + w_{-, im} |m^*\rangle \downarrow \otimes |-\rangle \right) \]

\[ D_{\mu \nu}^s = \sum_{\mathbf{k}} \sum_{m, n, p, q} C_{pq}^{mn} [j_{\mu, \uparrow}(\mathbf{k})]_{mn} [j_{\nu, \downarrow}(-\mathbf{k})]_{qp} \]
Extensions-Superfluid weight from linear response


\[ j_{\mu,\sigma}(k)_{mn} = \sigma \langle m | \partial_{\mu} \mathcal{H}_{\sigma}(k) | n \rangle_{\sigma} = \partial_{\mu} \varepsilon_{\sigma,m} \delta_{mn} + (\varepsilon_{\sigma,m} - \varepsilon_{\sigma,n})_{\sigma} \langle \partial_{\mu} m | n \rangle_{\sigma} \]

No geometric effect for single band systems
Inter-band processes important!

\[ D^s_{\mu\nu} = D^s_{\text{conv,}\mu\nu} + D^s_{\text{geom,}\mu\nu} \]

\[ D^s_{\text{geom,}\mu\nu} = \sum_{k} \sum_{m \neq n} C^m_{pq} [j_{\mu,\uparrow}(k)]_{mn} [j_{\nu,\downarrow}(k)]_{pq} \]
Extensions-Superfluid weight from linear response


Time reversal symmetry and uniform pairing:

\[ D_{\text{conv}, \mu \nu}^s = \sum_{k,m} \left[ -\frac{\beta}{2 \cosh^2 (\beta E_m/2)} + \frac{\tanh (\beta E_m/2)}{E_m} \right] \frac{\Delta^2}{E_m^2} \partial_\mu \varepsilon_m \partial_\nu \varepsilon_m \]

\[ D_{\text{geom}, \mu \nu}^s = 2\Delta^2 \sum_{k,m \neq n} \left[ \frac{\tanh (\beta E_m/2)}{E_m} - \frac{\tanh (\beta E_n/2)}{E_n} \right] \frac{\varepsilon_n - \varepsilon_m}{\varepsilon_n + \varepsilon_m - 2\mu} \text{Re} \langle \partial_\mu m|n\rangle \langle n|\partial_\nu m \rangle \]

In the limit of isolated but not necessarily flat band

\[ D_{\text{geom}, \mu \nu}^s = 2\Delta^2 \sum_k \frac{\tanh (\beta E_m/2)}{E_m} g_{\mu \nu}^m \]

Only single band quantities!

Question: How to get this using single band formalism?
Several complementary approaches

- Mean-field BCS theory
- Perturbative approach
- Numerics (ED, DMFT, DMRG)
Perturbative approach to flat band superfluidity


It is well-known that perturbation theory cannot describe the superconducting state since this has lower symmetry than the normal state (ideal Fermi gas or Landau-Fermi liquid)

In a flat band is the opposite: the normal state (insulator) has a much larger degeneracy than the superconducting one.

Interactions lift the degeneracy of the normal state

Perturbation theory can be used in this case!
We used perturbation theory in the form of Schrieffer-Wolff transformation.

For a recent review: S. Bravyi, D. P. DiVincenzo, and D. Loss, Ann. Phys. 326, 2793 (2011)

The SW transformation is a unitary transformation that decouples a chosen subspace and its orthogonal complement.
Effective Hamiltonian from the Schrieffer-Wolff transformation

\[ \hat{H}_{SW} = \hat{P}e^S\hat{H}e^{-S}\hat{P} \]

\[ \hat{P} \quad \text{Projector on the flat band many-body subspace} \]

SW effective Hamiltonian describes the low energy physics in terms of the degrees of freedom of the flat band renormalized by the interaction.

Perturbative expansion in powers of interaction strength over band gap \[ \frac{U}{E_g} \]

\[ \hat{H}_{SW} \approx \hat{H}_{\text{kin}}\hat{P} + \hat{P}\hat{H}_{\text{int}}\hat{P} + \hat{H}_{SW}^{(2)} + \ldots \]

*1st order:* Interaction term projected on the flat band many-body subspace

Exactness of BCS wavefunction at 1\textsuperscript{st} order

Pair creation operator

\[ \hat{b}_0^\dagger = \sum_j \hat{T}_j^\dagger = \sum_j \hat{d}_{j\uparrow}^\dagger \hat{d}_{j\downarrow}^\dagger = \sum_k \hat{f}_{k\uparrow}^\dagger \hat{f}_{-k\downarrow}^\dagger \]

Wannier w.f. Bloch w.f.

In a flat band the BCS wave function factorizes both in real and momentum space

\[ |\Omega\rangle = u^{N_c} \exp \left( \frac{v}{u} \hat{b}_0^\dagger \right) |\emptyset\rangle = \prod_j \left( u + vd_{j\uparrow}^\dagger d_{j\downarrow}^\dagger \right) |\emptyset\rangle = \prod_k \left( u + vf_{k\uparrow}^\dagger f_{-k\downarrow}^\dagger \right) |\emptyset\rangle \]

We show that the BCS wavefunction is an exact ground state (though possibly not the only one) if the Hamiltonian is time reversal symmetric and the uniform pairing condition is satisfied

\[ \hat{P} \hat{H}_{\text{int}} \hat{P} |\Omega\rangle = \left( \varepsilon_0 - \frac{n_\phi U}{2} \right) N |\Omega\rangle \]

The BCS is an exact wavefunction only if the \textit{uniform pairing condition} is satisfied

\[ n_\phi = \sum_i |W_\alpha(i)|^2 = \frac{V_c}{(2\pi)^d} \int_{B.Z.} d^d k |g_k(\alpha)|^2 \quad \forall \alpha \in S \]

\( S \) Subset of orbitals on which the flat band states, as well as the Cooper pair wavefunction, are nonzero

Implies the relation used already to derive the relation between quantum metric and superfluid weight

\[ \Delta_\alpha = \Delta = \text{const.} \]

Effective spin Hamiltonian

Retain in the projected interaction Hamiltonian only the terms that preserve the subspace defined by (meaning: all particles are paired)

\[\left(\hat{d}_{i\uparrow}^{\dagger}\hat{d}_{i\uparrow} - \hat{d}_{i\downarrow}^{\dagger}\hat{d}_{i\downarrow}\right)\left|\psi\right\rangle = 0\]

The result is a (pseudo)spin effective Hamiltonian

\[\hat{H}_{\text{spin}} = -U \sum_{i \neq j} J(|i - j|) \hat{T}_i \cdot \hat{T}_j\]

\[J(|i - j|) = \sum_{m\alpha} |W_{\alpha}(m - i)|^2 |W_{\alpha}(m - j)|^2\]

Pseudospin operators

\[\hat{T}_i^z = \frac{1}{2} (\hat{d}_{i\uparrow}^{\dagger}\hat{d}_{i\uparrow} + \hat{d}_{i\downarrow}^{\dagger}\hat{d}_{i\downarrow} - 1), \quad \hat{T}_i^+ = (\hat{T}_i^-)^\dagger = \hat{d}_{i\uparrow}^{\dagger}\hat{d}_{i\downarrow}\]
Flat band superconductivity and flat band ferromagnetism

Flat band ferromagnetism has been study extensively, especially by Mielke and Tasaki


We show that at first order in the coupling U flat band ferromagnetism and flat band superconductivity are related by a particle-hole transformation

$$\hat{d}_{1\downarrow} \rightarrow \hat{d}_{1\downarrow}$$

Under the transformation the pseudo-spin operators are mapped into the true spin operators

$$\hat{S}_i^z = \frac{1}{2} (\hat{d}_{i\uparrow}^\dagger \hat{d}_{i\uparrow} - \hat{d}_{i\downarrow}^\dagger \hat{d}_{i\downarrow}) , \quad \hat{S}_i^+ = (\hat{S}_i^-)^\dagger = \hat{d}_{i\uparrow}^\dagger \hat{d}_{i\downarrow}$$

Superfluid weight from the spin effective Hamiltonian

From the spin effective Hamiltonian we obtain that the superfluid weight is related to the overlap functional for Wannier functions

\[ [D_{s}^{(\text{spin})}]_{i,j} \propto \partial_{q_i} \partial_{q_j} F_{ov}(\mathbf{q} = 0) \]

\[ F_{ov}(\mathbf{q})[W] = - \sum_{i,j} \sum_{\alpha} |W_{\alpha}(l-i)|^2 |W_{\alpha}(l-j)|^2 e^{i\mathbf{q} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \]

This functional is related to the Marzari-Vanderbilt functional and thus to the quantum metric. The mean field result is recovered.

**Physical picture:** The local Hubbard interaction produces pair-hopping terms between overlapping Wannier functions. Cooper pairs have a finite mass!

These are terms of the form \( J(|i-j|) \hat{T}_i^+ \hat{T}_j^- \)

Example: Creutz ladder

- Two orbitals per unit cell
- Simple band structure: two flat bands
- Wannier functions are completely localized on four sites (plaquette states)


Drude weight of the Creutz ladder

In one dimension is more appropriate to talk about Drude weight rather than superfluid weight.


The line is the prediction of our analytic result.

Dots are obtained using Density Matrix Renormalization Group on a chain of finite length (L = 12).
Compressibility and second order effects

Using the SW transformation we can calculate the energy up to second order in the coupling $U$

$$\frac{E^{(2)}}{L} = \left( 2\varepsilon_0 - \frac{U}{2} - \frac{U^2}{8E_{\text{gap}}} \right) \nu + \frac{3U^2}{64E_{\text{gap}}} \nu^2$$

This is important in order to obtain a finite compressibility, which is diverging at first order.

Our analytic result agrees with the DMRG simulations.

Drude weight and winding number

The Creutz ladder possesses a nonzero topological invariant, the winding number

\[ \mathcal{W} = \frac{i}{\pi} \int_{\text{B.Z.}} dk \text{Tr} \left( \sigma^y \mathcal{P}_k \partial_k \mathcal{P}_k \right) \]

\[ \mathcal{P}_k = |g_k \rangle \langle g_k| \]

We show that the quantum metric is related to the winding number.
Therefore the Drude weight is bounded from below also by the winding number

\[ \mathcal{M} \geq \frac{\mathcal{W}^2}{\text{rank}(\mathcal{P}_k)} \quad \Rightarrow \quad D \geq \mathcal{W}^2 \]

Several complementary approaches

- Mean-field BCS theory
- Perturbative approach
- Numerics (ED, DMFT, DMRG)
Lieb lattice

As a case study we consider the Lieb lattice geometry


Three bands: two dispersive and one strictly flat band (with zero Chern number)

Staggered hopping coefficients open a gap between the flat band and the dispersive bands.

Isolated flat band
Lieb lattice: superfluid weight


$D_s$ on the flat band depends strongly on $U$

$D_s$ on dispersive bands roughly constant

$D_s$ for the trivial flat bands is non-zero and large!

$\nu$ Total filling ($1<\nu<2$ for the flat band)

Temperature is zero here
The large superfluid weight and its strong dependence on the interaction within the flat band is explained by the geometric superfluid weight contribution.
Haldane model: a prototype topological Hamiltonian

A hopping Hamiltonian with at least 2 bands (2 orbitals per unit cell) is necessary to access topologically nontrivial bands (nonzero Chern number)

\[ \hat{H} = -t_1 \sum_{\langle ik \rangle} (\hat{a}_i^\dagger \hat{b}_k + \text{H.c.}) - t_2 \left( \sum_{\langle\langle ij\rangle\rangle} e^{i\Phi_{ij}} \hat{a}_i^\dagger \hat{a}_j + \sum_{\langle\langle kl\rangle\rangle} e^{i\Phi_{kl}} \hat{b}_k^\dagger \hat{b}_l \right) + \Delta \sum_i \hat{a}_i^\dagger \hat{a}_i \]

**NN hopping**

**next NN hopping**

**sublattice energy difference**

\[ \Phi_{ij} = \Phi_{kl} = \Phi > 0 \]

for the next NN hoppings according to arrow directions

Breaks time-reversal symmetry, zero net magnetic flux

F. D. M. Haldane, PRL 61, 2015 (1986)

Haldane got the Nobel prize for this model, among other things
Kane-Mele and Haldane models

Band structure of both Kane-Male and Haldane models. Quasi-flat lowest band. The flatness ratio is approximately 6

Kane-Mele and Haldane Hubbard models: superfluid weight, MF vs. ED results

Kane-Mele and Haldane Hubbard models: BKT transition temperature

In 2D the superfluid weight sets the transition temperature of the Kosterliz-Thouless transition. For small $U$ it is similar to the Mean-Field BCS critical temperature.

Perspectives

- Superfluid properties of ultracold gases

- Are there known materials for which the geometric contribution to the superfluid weight is sizable?

- Possibly high-$T_c$ superconductors

- Design of novel superconducting materials (e.g. carbon-based)

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