where root edge connects \( u_i \) to its parent.

Then

\[
\mu_N(C^{(t_0)}) = Z^{-1}_N \sum_{i=1}^{M} \sum_{i_0=1}^{\infty} \frac{Z_{n_i \ldots n_M = N+M-1t_0}}{n_i \geq \frac{N+M-1t_0}{M}, n_i \leq A, i \neq i_0}
\]

\[
+ R(A, N, t_0)
\]

where \( A > 0 \) is fixed and the remainder \( R(A, N, t_0) \) can be estimated as follows:

\[
R(A, N, t_0) \leq M^2 w(C^{(t_0)}) \sum_{n_1, \ldots, n_M = N+M-1t_0}^{t_0, 1-M} \frac{Z_{n_1 \ldots n_M}}{Z_{n_1 \ldots n_M}^{1/3}} \left( \sum_{i=1}^{M} Z_{n_i}^{1/3} \right) \left( \sum_{i=1}^{M} Z_{n_i} \right)
\]

\[
\leq M^2 w(C^{(t_0)}) \sum_{n_2 > A}^{t_0, 1-M} C M^{3/2} Z_{n_2}^{-3/2} \cdot Z_0 \times Z_0^{-2}
\]

\[
\leq C_{t_0} \cdot A^{-1/2}.
\]

On the other hand,

\[
Z^{-1}_N \sum_{i=1}^{M} \frac{Z_{n_i \ldots n_M = N+M-1t_0}}{n_i \geq \frac{N+M-1t_0}{M}, n_i \leq A, i \neq i_0}
\]
$$\xi \xi_{0}^{1-M} \xi \xi_{0}^{1-M} \left( \sum_{k=1}^{A} \frac{Z_{N_{k}}}{Z_{N_{0}}} \xi_{0}^{N_{k}} \right) \xi \xi_{0}^{M-1}$$

as $N \to \infty$, which together with (3.4) proves the claim.

The existence of the limits (3.2) together with a precompactness property of $\{\mu_{N}\}$, which we omit here, implies by standard results (see e.g. Billingsley) that the limit $\mu$ exists, and $\mu(B(z_{0}))$ is given by (3.2).

**Single spine:** Given $A > 0$ and $z_{0}$ as before let

$$B_{A}(z_{0}) = \{ x \mid B_{A}^{c}(R, z_{0}) = \frac{2}{3} \}$$

and $(3.4)$ implies that

$$\lim_{N \to \infty} \mu_{N}(B_{A}(z_{0})) = C_{z_{0}} \cdot A^{\frac{1}{2}}$$

and hence $\mu(B_{A}(z_{0})) = C_{z_{0}} \cdot A^{\frac{1}{2}}$ and

$$\mu(A > 0) = 0$$

But $S_{0} = \bigcup_{A > 0} \lim_{N \to \infty} B_{A}(z_{0})$ and clearly $\mu(S_{0}) = 0$. 
Distribution of branches: Given \( R > 0 \) and a finite tree \( \tau_i \) with a marked leaf \( v_0 \) different from \( v \), and consider the simple path \( (v_0 = s_1, s_2, \ldots, s_n = w) \) as a finite spine of \( \tau_i \). For \( i \in \mathbb{N}_1 \), let \( \tau_{i,n} \) be the subtree spanned by \( s_0, \ldots, s_n \) and the branches rooted at \( s_1, \ldots, s_{n-1} \) with \( s_n \) marked. Define

\[
\mathcal{L}_n(\tau_{i,1}) = \{ \xi \in \mathcal{L} \mid \tau_{i,n}(\xi) = \tau_{i,1} \}.
\]

Then formula (3.5) implies that

\[
\mu(\mathcal{L}_n(\tau_{i,1})) = \sum_{v_0} \prod_{v \in \tau_{i,1} \setminus \{s_0, w\}} w_{v_0}
\]

which is equivalent to the content of 1) and 2) of Thm. 3.2.

Hausdorff dimension of \((\mathcal{L}, \mu)\)

Lemma 3.2: For a critical G-W tree we have

i) \( \langle |B(R, 5_0)| \rangle \) = \( R \)

ii) \( \mathbb{P}(\{ \xi \in J \mid h(\xi) > R \}) = \frac{2}{g''(1)} R + O(R^2) \)

Proof: i) Let \( H_i = \langle h_i(\xi) \rangle \). Then

\[
H_0 = H_1 = 1,
\]

\[
H_{i+1} = \sum_{n=1}^{\infty} (n-1) w_n Z_n H_i.
\]
\[ \sum_{n=0}^{\infty} n p_n \cdot H_i = H_i = 1. \]

Hence
\[ \langle 1 B(R, v_0) \rangle_{p_1} = \sum_{i=1}^{R} H_i = R. \]

It is a classical result of Kolmogoroff (1931) and can be found in Altschuler and Ney.

**Theorem 3.3** For the generic random tree \((Y, \mu)\) we have

\[ d_a = 2 \quad \text{and} \quad d_a = 2 \quad \text{almost surely}. \]

**Proof of \(d_a = 2\)** For \( z \in G \) the contribution to \(|B^z(R, v_0)|\) from a branch \( T \) rooted at \( v_i \) is \(|B^z(R-i, v_i)|\). Since branches are independent, this gives

\[ \langle 1 B(R, v_0) \rangle_{p_i} = \sum_{i=1}^{R-1} \frac{R}{11 \cdot \prod (k_j^{j})} \left( 1 + \sum_{j=1}^{R} \frac{1}{1 + k_j^1 + k_j^2} \right) + R \]

\[ = \frac{1}{2} g''(1) R (R-1) + R. \]

That \(d_a = 2\) a.s. follows by a closer analysis of the generating function \( g \) (see e.g. DJW2).

**Theorem 3.4** For the generic random tree \((Y, \mu)\) we have

\[ d_s = \frac{4}{3} \quad \text{and} \quad d_s = \frac{4}{3} \quad \text{a.s.}. \]
Proof of $\bar{d}_d \leq \frac{\gamma}{3}$ and $d_d \geq \frac{\gamma}{3}$ a.s.

That $d_d \geq \frac{\gamma}{3}$ a.s. is a consequence of

Thm. 2.1 and Thm. 3.3.

To show $\bar{d}_d \leq \frac{\gamma}{3}$ we need two simple

lemmas.

Lemma 3.3 For any finite tree $T \in \mathcal{T} \setminus \mathcal{T}_a$ and

$0 \leq x \leq 1$ we have

$$P_T(x) \geq 1 - |T|/x.$$  \(\text{Proof:} \) Let $T_1, \ldots, T_n$ be the sub-tree of

$T$ rooted at the vertex $v_i$ next to the root.

By (a generalization) of Lemma 1.2 we have

$$P_T(x) = \frac{1-x}{n - \sum_{i=1}^{n-1} P_{T_i}(x)}.$$  \(\text{If} \) $|T| = 1$ then $P_T(x) = 1 - x$, and the

lemma follows by induction on $|T|$. \(\Box\)

Lemma 3.4 Let $x \in \mathcal{S}$. For all $L \geq 1$ and

$0 \leq x \leq 1$ we have

$$P_{e_L}(x) \geq 1 - \frac{L}{L - Lx - \sum_{T \in \mathcal{T}} \frac{\bar{d}_d}{|T|} (1 - P_T(x))},$$  \(\text{where} \) $\sum_{T \in \mathcal{T}}$ indicates the sum over all branches

of $\mathcal{T}$ rooted at vertices $s_1, \ldots, s_L$ on the spine.

\(\text{Proof:} \) Let $P_{e_L}(x)$ denote the contribution to

$P_e(x)$ from paths not hitting $s_{L+1}$. It
suffices to show the inequality for $P_t^L(x)$. 

For $L = 2$, it obviously holds since $P_2^L(x) > 0$.

Assume it holds for $L = 1$ and use Lemma 1.2 to write

$$P_L(x) = \frac{1 - x}{n - P_{T_1}^L(x)} = \frac{1}{n - P_{T_1}^L(x)} \sum_{k=1}^{n-2} P_{T_k}^L(x)$$

where $T_1, \ldots, T_{n-2}$ denote the finite branches of $T$ rooted at $v_1$ and $v_2$, the infinite branch, and $m = 0$. Together with the induction hypothesis, this gives

$$P_L(x) = \frac{1 - x}{1 + (1 - P_{v_1}^L(x)) + \sum_{k=1}^{n-2} (1 - P_{T_k}^L(x))}$$

$$\geq \frac{1 - x}{1 + \frac{1}{L-1} + \frac{1}{L-1} x + \sum_{T \in \mathcal{X}} (1 - P_T^L(x))}$$

$$\geq \frac{\frac{L-1}{L}}{1 + \frac{1}{L-1} x + \sum_{T \in \mathcal{X}} (1 - P_T^L(x))}$$

$$\geq \left(1 - \frac{1}{L}\right)(1 - x) - \left(1 - \frac{1}{L}\right)x - \sum_{T \in \mathcal{X}} (1 - P_T^L(x))$$

$$\geq \left(1 - \frac{1}{L}\right)(1 - x) - (L-1)x - \sum_{T \in \mathcal{X}} (1 - P_T^L(x))$$

$$\geq \frac{1 - \frac{1}{L}}{L - 1} - \frac{\sum_{T \in \mathcal{X}} (1 - P_T^L(x))}{L - 1}$$

Back to the proof of $\bar{d}_s \leq \frac{4}{3}$;
Let \( v \) be any vertex on the spine having \( k \) left branches and \( l \) right branches. The probability \( q_R \) that at least one of those branches has height \( > R \) fulfills

\[
q_R \leq (k+l) \left( \frac{2}{g^{u}(1)R} + O(R^{-2}) \right)
\]

by Lemma 3.2 (i). Hence the \( \mu \)-probability \( q_R \) that a branch at \( v \) has height \( > R \) fulfills

\[
q_R \leq (\frac{2}{g^{u}(1)R} + O(R^{-2})) \sum_{l,k \geq 0} (k+l) \varphi(l,k)
\]

where

\[
\varphi(l,k) = \mathbb{W}^{l+k+2} \mathbb{E}_{0} \cdot \mathbb{E}_{0} = \mathcal{P}(l+k+1)
\]

by Thm. 3.2. It follows that the last sum is \( g^{u}(1) \) such that

\[
q_R \leq \frac{2}{R} + O(R^{-2}).
\]

Using independence of branches one gets that the probability that all branches rooted at spine vertices \( A_1, \ldots, A_R \) have height \( \leq R \) fulfills

\[
\mu(\mathcal{B}_R) = (1-q_R)^R \geq a > 0,
\]

where \( a \) is indep. of \( R \). Denoting by \( \xi \) the expectation value of \( \mu \) conditioned on \( \mathcal{B}_R \) this gives
\[
\langle Q_\tau(x) \rangle_\mu \geq a^{-1} \langle (1 - P_\tau(x))^{-1} \rangle_R
\]
\[
\geq a^{-1} \left( \left( \frac{1}{R} + R x + \sum_{T \subseteq T_c}^{\leq R} \langle 1 - P_T(x) \rangle \right)^{-1} \right)_R
\]
\[
\geq a^{-1} \left( \frac{1}{R} + R x + \sum_{T \subseteq T_c}^{\leq R} \langle 1 - P_T(x) \rangle \right)_R
\]
\[
\geq a^{-1} \left( \frac{1}{R} + R x + \sum_{T \subseteq T_c}^{\leq R} \langle 1 + 1 \rangle \right)_R
\]

Since \( T \) in the last average has height \( \leq R \), it follows that
\[
\left( \sum_{T \subseteq T_c}^{\leq R} \langle 1 + 1 \rangle \right)_R \leq \langle (1 - q_T) \rangle_R \sum_{i=1}^{R} \langle B_{t_i}^i(R, s_i) \rangle_\mu
\]

where \( B_{t_i}^i(R, s_i) \) denotes the subtree of \( T \) spanned by the vertices in the branches rooted at \( s_i \) at distance \( \leq R \) from \( s_i \). Using independence of branches we get that \( \langle B_{t_i}^i(R, s_i) \rangle_\mu \) is independent of \( i \) and given by
\[
\langle B_{t_i}^i(R, s_i) \rangle_\mu = \sum_{k \in \mathbb{Z}} g_{k+e} \mathbb{Q}(k, e) \langle B_t(R, k) \rangle_\mu
\]
\[
= g' R^k \mu(R)
\]

We conclude that, since \( q_T \to 0 \) as \( R \to \infty \), we for \( R \) large:
\[
\langle Q_\tau(x) \rangle_\mu \geq C' \left( \frac{1}{R} + R x + g'' R \right)^{-1}
\]

Setting \( R = \lceil x^{-\frac{1}{3}} \rceil \) yields \( \langle Q_\tau(x) \rangle_\mu \geq C x^{-\frac{1}{3}} \) and hence \( d_+ \leq x^{-\frac{1}{3}} \).
Remark on the two-point function, mass, and mass exponent for generic random trees.