Recurrence properties of random infinite graphs.

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0. Introduction

- Not Renyi–Erdos's random graphs
- Local weights, finite size limits:
  a) Two-dim quantum gravity (F.D.J. Zinn-Justin: Quantum Gravity, 1996)
  b) Percolation (G. R. Grimmett: Percolation, 1999)

1. Recurrence, spectral dim. and Hausdorff dim.

**Graph** \( G = (V, E) \), \( V = \) vertex set, \( E = \) edge set, unoriented, countable, locally finite

**Size** \( |G| = |E(G)| \)

Degree of \( \nu \in V(G) = \#(\text{edges containing } \nu) = \sigma_\nu \)

**Path**; sequence of different edges \((v_0, v_1), (v_1, v_2), \ldots, (v_{b-1}, v_b)\)

**Connected graph**; Any pair of vertices are connected by a path. Assumed in the following.

**Graph distance**:
\[ d_G(v, v') = \text{minimal size of a path connecting } v \text{ and } v' \]

**Planar graph**

**Walk**; \( w: v \rightarrow v' \) is a sequence of edges \((v_0, v_1), (v_1, v_2), \ldots, (v_{b-1}, v_b)\), \( v_0 = v \), \( v_b = v' \)

**Simple random walk** on \( G \):
\[ p_G(w) = \frac{1}{|G|} \sum_{i=0}^{n} \sigma_G(v_i) \]
where $w(i)$ is the $i$'th vertex on $w$.

Defines a probability distribution $p_\alpha^n$ on walks $w$ of length $n$ originating at some fixed $v_0$. Also defines a probability distribution $p_\alpha^\infty$ on infinite walks originating at some fixed $v_0$ by

$$p_\alpha^\infty(\{w: v_0 \to x \mid w(i) = \omega_0(i), \ i = 0, \ldots, n\})$$

$$= p_\alpha^n(\omega_0)$$

where $w: v_0 \to x$ denotes an arbitrary infinite walk originating at $v_0$ and $\omega_0$ is any given walk of length $n$ originating at $v_0$.

**Def. 1.1** $\alpha$ is recurrent if, given $v_0$, we have

$$p_\alpha^\infty(\{w: v_0 \to x \mid \exists i > 0 : w(i) = v_0\}) = 1,$$

or

$$\sum_{n=1}^{\infty} p_\alpha^\infty(A_n) = 1$$

where

$$A_n = \{w: v_0 \to x \mid w(n) = v_0, w(i) \neq v_0 \text{ for } 1 \leq i \leq n-1\}.$$  

$$p_\alpha^\infty(A_n) = p_\alpha^\infty(\{w: v_0 \to v_0 \mid w(i) \neq v_0 \text{ for } 1 \leq i \leq n-1\})$$

**Lemma 1.1** Let

$$B_n = \{w: v_0 \to x \mid w(n) = v_0\}.$$  

Then $\alpha$ is recurrent if and only if

$$\sum_{n=1}^{\infty} p_\alpha^\infty(B_n) = \infty.$$
Proof. Let \( t_k \) denote the time of \( k \)th return to \( v_0 \), \( k = 0, 1, 2, \ldots \). Then

\[
P_a^{\infty} (B_n) = \sum_{l=0}^{n} P_a^{\infty} (i_l = n)
\]

\[
= \frac{1}{\beta} \sum_{l=0}^{n} \sum_{1 \leq n_1 < n_2 < \ldots < n_l < n} P_a^{\infty} (i_1 = n_1, \ldots, i_l = n_l, i_{l+1} = n)
\]

\[
= \frac{1}{\beta} \sum_{l=0}^{\infty} \sum_{1 \leq n_1 < \ldots < n_l < n} \frac{1}{l!} P_a^{\infty} (A_{n_l} - n_{l-1})
\]

where \( n_0 = 0 \). Summing over \( n \) gives

\[
\sum_{n=0}^{\infty} P_a^{\infty} (B_n) = \sum_{l=0}^{\infty} \left( \sum_{m=1}^{\infty} P_a^{\infty} (A_m) \right)^l
\]

which proves the claim. \( \qed \)

Example 1.1

a) Any finite graph is recurrent; it is easy to see that \( \sum_{n=1}^{\infty} P_a^{\infty} (B_n) = \infty \).

b) \( \mathbb{Z}^d \) is recurrent if \( d = 1, 2 \) and not recurrent (transient) if \( d \geq 3 \).

Use Fourier transform now: Set

\[
G (k) = \sum_{v \in \mathbb{Z}^d} \sum_{w : v \rightarrow v} P_a (w) e^{-i k (v-v_0)}
\]

\[
= \sum_{v \in \mathbb{Z}^d} \sum_{w : v \rightarrow v} \prod_{i=0}^{\omega - 1} (2d)^{-1} e^{-i k (w(i+1) - w(i))}
\]

\[
= \sum_{n=0}^{\infty} \left( (2d)^{-1} \sum_{\omega=1}^{\omega - 1} (e^{-ik \alpha} + e^{ik \alpha}) \right)^n
\]
\[
(1 - d^{-1} \sum_{\kappa = 1}^{d} \cos k\kappa)^{-1}, \quad -\pi \leq k\kappa \leq \pi.
\]

Integrating \( G(k) \) w.r.t. \( k \) over \([-\pi, \pi] \) it is seen that only \( \nu = \nu_0 \) contributes with the result
\[
(2\pi)^d \sum_{\omega: \nu_0 \rightarrow \nu} P_0^a(\omega) = (2\pi)^d \sum_{n=1}^{\infty} P_0^a(B_n). \]

On the other hand it is seen that \( G(k) \) is integrable if and only if \( d \geq 2 \).

**Def. 12.** (Generating fcts. for return probabilities.)

\[
P_0^a(x) = \sum_{n=1}^{\infty} P_0^a(A_n)(1-x)^{\frac{1}{2}}
\]

\[
Q_0^a(x) = \sum_{n=0}^{\infty} P_0^a(B_n)(1-x)^{\frac{1}{2}}
\]

Note
\[
Q_0^a(x) = \frac{1}{1 - P_0^a(x)}
\]

and \( G \) is recurrent if and only if \( Q_0^a(x) \) is divergent at \( x = 0 \):

\[
Q_0^a(x) \sim x^{-\alpha} \quad \text{as} \quad x \to 0
\]

\[
1 - P_0^a(x) \sim x^{\alpha} \quad \text{as} \quad x \to 0.
\]

**Spectral dimension of** \( G \):

\[
d_s = 2(1-\alpha), \quad \text{if} \quad \alpha > 0
\]

\[
\alpha = 0: \quad Q_0^a(x) \sim |\log x|^{\beta}, \quad \beta > 0, \quad d_s = 2.
\]
Ex. 1.2 \( G = Z_+ \), \( P_{G_1}(x) = P_{G_2}(x) \)

\[
P_{G_1}(x) = \frac{\frac{1}{2}(1-x)}{1 - \frac{1}{2} P_{G_2}(x)}
\]

\( \Rightarrow \) \( P_{G_1}(x)^2 - 2P_{G_1}(x) + (1-x) = 0 \)

\( \Rightarrow \) \( P_{G_1}(x) = 1 - \sqrt{x} \)

i.e. \( x = \frac{1}{2} \), \( d_5 = 1 \).

Lemma 1.2 Let \( G_1 \) and \( G_2 \) be two connected disjoint graphs, \( v_1 \in V(G_1), v_2 \in V(G_2) \), and let \( G_0 \) be obtained by identifying \( v_1 \) and \( v_2 \) and adding a new vertex \( v_0 \) as well as the link \( (v_0, v_1) = (v_0, v_2) \). Then

\[
P_{G_0}(x) = \frac{1-x}{1 + \sigma_{v_1} + \sigma_{v_2} - P_{G_1}(x) - P_{G_2}(x)}
\]

Proof.

Ex. 1.3 (Comb with infinite teeth.)

Here \( G_1 = Z_+ \) and \( G_2 = C \), \( \sigma_{v_1} = \sigma_{v_2} = 1 \).
\[
\begin{align*}
P_c(x) &= \frac{1-x}{3 - P_0(x) - P_c(x)} = \frac{1-x}{2 + \sqrt{x} - P_c(x)} \\
\Rightarrow \quad P_c(x) &= 1 - x^{\alpha} \sqrt{1 + \frac{\alpha}{2} \sqrt{x}} + \frac{1}{2} \sqrt{x},
\end{align*}
\]

i.e. \( \alpha = \frac{1}{4} \) and \( \frac{\alpha}{2} = \frac{3}{2} \).

Hausdorff dimension \( \mathcal{H} \) connected, \( V_0 \in V(G) \).

Let \( \mathcal{B}(R, V_0) \) be the ball of radius \( R \) centered at \( V_0 \), i.e. \( \mathcal{B}(R, V_0) \) is the subgraph of \( G \) spanned by vertices at graph distance \( \leq R \) from \( V_0 \). Then

\[
d_H = \lim_{R \to \infty} \frac{\log |V(\mathcal{B}(R, V_0))|}{\log R}
\]

if it exists.

Ex. 1.4 (a) \( d_H = 0 \) if \( G \) is finite,

(b) \( d_H = d \) if \( G = \mathbb{Z}^d \),

(c) \( d_H = 1 \) if \( G = \mathbb{Z}^+ \),

(d) \( d_H = 2 \) if \( G = \mathbb{C} \). Note: \( d_H \neq d_H^c \),

(e) \( d_H = \infty \) if \( G \) is a Cayley graph.

Note: in this case Lemma 2 gives

\[
P_c(x) = \frac{1-x}{3 - 2 P_c(x)}
\]

\[
\Rightarrow 2 P_c(x)^2 - 3 P_c(x) + 1-x = 0
\]

\[
\Rightarrow P_c(x) = \frac{3 - \sqrt{9 - 8(1-x)}}{4} = \frac{3}{4} - \frac{1}{4} \sqrt{1 + 8x}
\]

Hence \( P_c(0) = \frac{1}{2} < 1 \). \( G \) not recurrent.
Relation between $d_s$ and $d_a$.

Let $G$ be a graph and $G_0$ a subgraph of $G$. By $\tilde{G}_0$ we denote the subgraph spanned by the vertices of $G_0$, having at least one neighbour in $V(G) \setminus V(G_0)$, and by $\bar{G}_0$ we denote the subgraph spanned by $G_0$ and nearest neighbours of vertices in $G_0$. For $v \in V(G_0)$ we define the out-degree $\sigma_{0,v}$ of $v$ as the number of neighbours of $v$ in $V(G) \setminus V(G_0)$.

Lemma 1.3 Let $G$ be a connected graph and $G_0$ a subgraph such that $\tilde{G}_0 \neq \emptyset$. Setting

$$q_0(w) = \frac{1}{\sigma_{0,v}} \sigma_{0,w}^{-1}$$

for a walk in $G$ we have, for arbitrary $v_0 \in G_0$, that

$$\sum_{v \in \tilde{G}_0} \sum_{w : v_0 \rightarrow v} q_0(w) \sigma_{0,v} \leq 1,$$

with equality holding if $\bar{G}_0$ is connected and recurrent.

Proof We claim that the LHS is the probability that a walk $w$ starting at $v_0$ leaves $G_0$. Given such an $w$ let $v \in \tilde{G}_0$ denote the last vertex visited in $G_0$ before