Part II. The Ising model

A graph connected graph, with set $V$, edge set $E$

$$J = \{ \omega \in E \}$$

position edge weights

A spin configuration on $G$ is an element $\sigma \in \{ \pm 1 \}^V$

The energy of $\sigma \in \{ \pm 1 \}^V$ is $H(\sigma) = -\sum_{\omega \in E} J_\omega \sigma_\omega$ (lower if neighboring spins are aligned)

This induces a probability measure $\mu_\beta$ on $\{ \pm 1 \}^V$:

$$\mu_\beta(\sigma) = \frac{1}{Z_\beta} e^{-\beta H(\sigma)}$$

where

$$Z_\beta = \sum_{\sigma \in \{ \pm 1 \}^V} e^{-\beta H(\sigma)}$$

is the partition function for the Ising model on $(G,J)$.

and $\beta = \frac{1}{T} > 0$ is the inverse temperature.

Questions:
1. Can we compute efficiently the partition function?
2. How does the system evolve when $T$ varies?

We shall first make the statement at $T = \text{max}$, then answer 2, then answer 1 for bipartite graphs.

II.1. Phase transition

So, how does the pdf measure $\mu_\beta$ depend on $T$? Let us begin with extreme cases:

- $T \to 0$: all $\sigma$ is equiprobable (i.e. on $n$, Bernoulli variables).
- $T \to \infty$: $\beta = 0 \Rightarrow$ only $\sigma$'s have positive $\sigma = \pm 1$.

For $G$ finite, energy varies smoothly $\beta = \frac{1}{T} \in (0,\infty)$.

But now, imagine $G$ finite $\Rightarrow G$ infinite graph.

Example: $G = \text{square lattice} \times \mathbb{T}^2$ $J = 1$ (see: $\beta = 0.45$, $p = 1-e^{-4}$ in Fig. 1: square layer at $\beta = 0.55$).
Configurations typiques du modèle d’Ising en dimension 2 avec condition au bord périodique ($N = 500$), pour différentes valeurs du paramètre $p_\beta \equiv 1 - e^{-2\beta}$.
"Experimental observation" of a phase transition: a sharp qualitative change in the model at some critical $\beta_c$. C(1973)

Here is one possible rigorous definition of $\beta_c$ (among many):

Let $G_n$ be a $\mathcal{G}$-subgraph,

Let $\mathcal{M}_n$ denote the Dirichlet measure on $G_n \times \mathcal{G}$ conditioned with $+$-boundary condition.

**Standard fact (follow from FKG inequalities):** the measure $\mathcal{M}_{\beta_n} \rightarrow \mathcal{M}_{\beta}$ ($\beta \rightarrow \infty$) is the unique $+$-Dirichlet measure of the chain of $G_n \times \mathcal{G}$.

In particular, for any real $\nu \in \mathbb{R}$, we have an expectation

$$\langle \sigma_0 \rangle_{\beta_n}^+ = \lim_{n \rightarrow \infty} \langle \sigma_0 \rangle_{\beta_n}^+$$

**Fact (GKS inequalities):** $\beta \rightarrow \langle \sigma_0 \rangle_{\beta_n}^+$ is non-increasing.

Clearly, $\langle \sigma_0 \rangle_{\beta_n}^+$ is defined $\beta_c := \text{sup} \{ \beta > 0 | \langle \sigma_0 \rangle_{\beta_n}^+ > 1 \} \in [0, \infty)$. (only depends on $(G, \mathcal{G})$, not on $G_n \times \mathcal{G}$, nor on $\nu$)

**Examples:**

- $\mathcal{G}$ finite $\Rightarrow \beta_c = 0$, no phase transition.
- $\mathcal{G} = \mathbb{Z} \times \mathbb{Z}$ $\Rightarrow \beta_c = \infty$ (Ising's PhD, 1925), no phase transition.

**Fact:** (Peierls, 1936): For a large class of subcritical weighted graphs $(G, \mathcal{G})$, including all planar bipartite graphs (with low $\beta$-values), $0 < \beta_c < \infty$.

**Question:** How can one determine $\beta_c$?

We shall start with a heuristic equation for $\beta_c$. 
### 2. Kramer-Wannier Duality

#### A. High-Temperature Representation

Let \((G, J)\) be an oriented finite abstract weighted graph.

\[
\begin{align*}
\mathcal{Z}_\beta^J(G) &= \sum_{\sigma} e^{-\beta \sum E_{\sigma}} = \sum_{\sigma} \prod_{e \in \text{edges}(\sigma)} e^{\beta J_e \sigma_e} \\
&= C \sum_{x \in \text{colors}} \prod_{e \in \text{edges}} \sigma_e^{x_e} = C \sum_{x \in \mathcal{C}(G)} \prod_{e \in \text{edges}} (1 + x_e \sigma_e) \\
&= C \sum_{x \in \mathcal{C}(G)} \prod_{e \in \text{edges}} \sigma_e^{x_e} (1 + x_e) \\
&= C \sum_{x \in \mathcal{C}(G)} \prod_{e \in \text{edges}} (1 + x_e \sigma_e) \\
\end{align*}
\]

where \(\mathcal{C}(G) = \{ x \in \{0, 1\}^E \mid \text{deg}_G(v) \equiv \text{color}_G(v) \mod 2 \} \)

\[\Rightarrow \mathcal{Z}_\beta^J(G) = C \sum_{x \in \mathcal{C}(G)} x^\Gamma, \quad \text{where} \quad x^\Gamma = \prod_{e \in \text{edges}} x_e, \quad x_e = \exp(-\beta J_e) \quad (\forall G \text{ finite bipartite graph})\]

#### B. Low-Temperature Representation

Now, assume \((G, J)\) is a planar weighted graph, and let \(G^*\) be its dual graph.

To \(\sigma \in \mathcal{E}(G)\), one can associate \(\gamma(\sigma) = \{ e^\sigma \in \mathcal{E}(G^*) \mid x^\Gamma = \prod_{e \in \text{edges}} x_e, \quad x_e = \exp(-\beta J_e) \}

This map \(\mathcal{E}(G) \rightarrow \mathcal{E}(G^*)\) is clearly a bijection: \(\gamma(\sigma) = \gamma(\bar{\sigma})\).

By definition, it maps is \(\gamma(\sigma)\) to \(\gamma(\bar{\sigma})\). A boundary of \( \sigma \) is \( \text{ indiv}(C_1 \rightarrow C_2) = \text{ Ker}(C_1 \rightarrow C_2) = \mathcal{E}(G^*)\).

So we have a bijection \(1 \leftrightarrow 1\) map \(\mathcal{E}(G) \rightarrow \mathcal{E}(G^*)\).

Furthermore, \(\mathcal{H}(\sigma) = \sum_{e \in \text{edges}(\sigma)} J_e \sigma_e = \sum_{e' \in \text{edges}(\sigma)} J_{e'} - \sum_{e'' \in \text{edges}(\sigma)} x_{e''} \sum_{e' \in \text{edges}(\sigma)} J_{e'} - \sum_{e'' \in \text{edges}(\sigma)} \sigma_{e''}\)

\[\Rightarrow e^{-\beta \mathcal{H}(\sigma)} = e^{\beta \sum_{e \in \text{edges}(\sigma)} J_e \sigma_e} \prod_{e' \in \text{edges}(\sigma)} \exp(-\beta J_{e'}) \]

\[\Rightarrow \mathcal{Z}_\beta^J(G) = C \sum_{x \in \mathcal{C}(G^*)} x^\Gamma, \quad \text{where} \quad x^\Gamma = \prod_{e \in \text{edges}} x_e, \quad x_e = \exp(-\beta J_e) \]
Duality

Note that the weights \( x_0 = \text{tan}(\beta \pi) \in [0,1] \) and \( x_0 = \exp(-2\beta \pi) \in [0,1] \) are related by \( x + x^* + x x^* = 1 \).

(Dual weight). Equating both equations above and varying all the constants, one gets:

\[
2^{1/2} \prod_{i=0}^{n-1} \sum_{x(i)} = 2^{1/2} \prod_{i=0}^{n-1} \sum_{x'(i)}
\]

Furthermore, setting \( J = 1 \) (\( \Rightarrow x = \text{tan}(\beta) \), \( x^* = e^{-2\beta} \)) and choosing \( \beta^* = \text{tan}(\beta^*) = e^{-2\beta} \), we get:

\[
f_g(\beta) = \lim_{n \to \infty} \frac{1}{\log n} \log \mathbb{E}_n(x) = (\text{analytic part of } \beta) + \lim_{n \to \infty} \frac{1}{\log n} \log \left( \sum_{x' \in E_n} x'(i) \right)
\]

\[
f_g^*(\beta) = \text{analytic part of } \beta
\]

This relates the free energy of \( G \) to the free energy of \( G^* \).

Now, the relation \( \text{tan}(\beta^*) = e^{-2\beta} \) and \( \beta^* \) small to \( \beta \) by good use work,

Thus, \( \beta^* \approx \frac{1}{2} \log (1 + \pi) \approx 0.441 \ldots \)

Consider the case \( G = \# \), \( J = 1 \). Then, \( G^* = G \Rightarrow f_g(\beta^*) = \text{analytic} + f_g(\beta) \).

Since \( f_g \) is non-analytic at \( \beta_c \) (non), assuming this is the only singularity, we must have \( \beta_c = \beta^* \).

(Kramers-Wannier, 1941).

Rigorous proof: Onsager, 1944.

In the rest of the class, we will:

- show how to compute efficiently the high-loop expansion for any finite graph.
- extend the K-1 duality displayed above to non-planar graphs.
- determine \( \beta_c \) for any bipartite weighted planar graph.
- show that \( f_g \) is analytic at \( \beta_c \) for any such graphs.